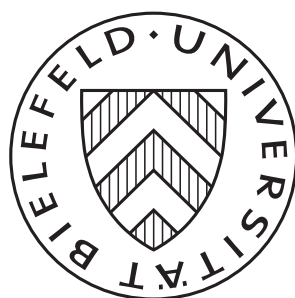


First-price Sealed-bid Auctions with Smoothly Ambiguity-Averse Bidders

Tianyu Ma and Frank Riedel



First-price Sealed-bid Auctions with Smoothly Ambiguity-Averse Bidders*

Tianyu Ma and Frank Riedel

Center for Mathematical Economics, Bielefeld University

August 6, 2026

Abstract

We analyze first-price sealed-bid auctions with independent private values in which bidders are uncertain about the distribution of their opponents' valuations and have smooth ambiguity preferences. We characterize the unique non-decreasing symmetric equilibrium, whose bidding function solves a nonlinear ordinary differential equation with an endogenous ambiguity-adjusted distribution. Bids increase in ambiguity aversion, lie between the Bayesian benchmarks generated by the least and most competitive candidate priors, and converge to the maxmin benchmark. Under constant relative ambiguity aversion (CRAA), the equilibrium bidding function is explicit.

We also compare auction formats. Under an ex ante predictive criterion, the second-price auction dominates under ambiguity neutrality, but sufficiently strong ambiguity aversion reverses the ranking. Under a model-based criterion, the first-price auction performs better in less competitive environments and worse in more competitive ones, while compressing the range of possible revenues. Bidder preferences also depend on ambiguity attitudes: CRAA bidders weakly prefer the second-price auction, whereas bidders with increasing absolute ambiguity aversion weakly prefer the first-price auction.

Keywords: First-price auctions, smooth ambiguity, ambiguity aversion, Knightian uncertainty, auction design, revenue comparison

JEL Classification: D44, D81, D82, C72

*The authors thank Gerrit Bauch, Victor Gonzalez-Jimenez, Dominik Karos, Bernhard Kasberger, Lasse Mononen, Lars Müller, Fernando Payró Chew, Jurek Preker, Guillaume Pommey and Jean-Marc Tallon for their helpful questions, comments, and discussions. Financial support by the German Research Foundation (DFG) [RTG 2865/1 – 492988838] is gratefully acknowledged.

1 Introduction

Auctions are ubiquitous mechanisms for allocating goods and contracts, with applications ranging from spectrum sales and government procurement to online marketplaces. Traditional auction theory typically assumes that bidders hold precise probabilistic beliefs about their rivals' valuations and choose bids to maximize expected utility. In practice, however, bidders often possess only incomplete or imprecise statistical information about competitors' valuations. They therefore face ambiguity, or Knightian uncertainty, about the underlying distribution of opponents' values. This paper studies first-price sealed-bid auctions in which bidders face such ambiguity and have smooth ambiguity-averse preferences in the sense of [Klibanoff, Marinacci, and Mukerji \(2005\)](#).

Ambiguity matters in first-price auctions because bidding requires a strategic trade-off that depends sensitively on beliefs about rivals' valuations. In a first-price auction, a bidder must shade her bid below her valuation, balancing the gain from bidding more aggressively against the risk of paying too much conditional on winning. When the distribution of opponents' valuations is itself uncertain, this trade-off is no longer governed by a single probability model. Ambiguity can therefore alter both the level of bids and the extent of bid shading, with direct consequences for allocation, bidder surplus, and seller revenue. Since first-price auctions require bidders to act strategically under uncertainty about others' behavior, they provide a natural environment in which to study the effects of ambiguity-averse preferences.

We establish existence and uniqueness of a symmetric interim equilibrium under suitable regularity conditions. The associated equilibrium bidding function satisfies a nonlinear ordinary differential equation (ODE) that extends the standard bidding equation from the Bayesian model. In the Bayesian benchmark, bid shading is governed by the reverse hazard rate of the distribution of opponents' valuations. Under ambiguity, this object is replaced by an ambiguity-adjusted average of reverse hazard rates across models.

The contrast with the Bayesian benchmark becomes particularly sharp under constant relative ambiguity aversion, where the equilibrium bidding function can be derived explicitly. In this case, bidders bid as if opponents' valuations were drawn from a geometrically weighted average of the candidate priors. This effective prior is more competitive, in the sense of first-order stochastic dominance, than the predictive prior obtained by averaging the candidate beliefs according to the second-order distribution. Consequently, ambiguity leads to more aggressive bidding than under ambiguity neutrality.

Motivated by this result, we derive general comparative statics for equilibrium bidding under ambiguity. In particular, we show that equilibrium bids lie between the

Bayesian equilibrium bids associated with the least competitive and the most competitive priors among the candidate models. More generally, equilibrium bidding is increasing in ambiguity aversion. Finally, we show that, in the limit, equilibrium bids converge to the maxmin expected utility benchmark characterized by [Lo \(1998\)](#).

Ambiguity also has interesting implications for comparative statics with respect to the number of bidders. In the Bayesian benchmark, greater competition in the form of additional bidders leads to more aggressive bidding. Under ambiguity, by contrast, this monotonicity need not hold. When bidders are sufficiently close to ambiguity neutrality, adding bidders may even reduce equilibrium bids, because for a fixed degree of ambiguity aversion the effective prior can become less competitive as the number of bidders increases.

We then explore the implications of ambiguity for auction-format comparison. From the seller’s perspective, there are two natural ways to rank auctions. When the seller faces the same Knightian uncertainty as the bidders, it is natural to assume that she evaluates ex ante expected revenue under the predictive prior, obtained by averaging candidate models according to the common second-order prior. Under this predictive criterion, the SPA yields higher revenue when bidders are ambiguity-neutral, but sufficiently ambiguity-averse bidders bid aggressively enough that the FPA eventually generates higher expected revenue.

Alternatively, the seller may be better informed about the underlying model, or one may wish to evaluate revenue separately under each candidate model. Under this model-by-model criterion, the seller prefers the FPA for less competitive models and the SPA for more competitive ones. Moreover, ambiguity compresses the range of possible seller revenues across candidate models under the FPA relative to the SPA.

Finally, we compare the two formats from the bidders’ perspective and show that smooth ambiguity does not in general yield a uniform ranking: under constant relative ambiguity aversion, bidders weakly prefer the SPA, whereas under increasing absolute ambiguity aversion they weakly prefer the FPA.

We now turn to the related literature. A natural benchmark is [Lo \(1998\)](#), who studies first-price auctions under the maxmin expected utility model of [Gilboa and Schmeidler \(1989\)](#) and shows that bidders behave according to the worst-case prior. [Chen, Katuščák, and Ozdenoren \(2007\)](#) use the α -MEU model to disentangle ambiguity from ambiguity attitudes, albeit under fairly restrictive assumptions on the set of priors. [Gretschko and Mass \(2024\)](#) study worst-case equilibrium under coarse distributional information, while [Kasberger and Schlag \(2024\)](#) derive minimax-loss bids without imposing equilibrium behavior. [Kim \(2024\)](#) provides comparative statics under maxmin preferences and shows that bidders bid more aggressively as the set of priors expands. The dissertation by [Carvalho \(2012\)](#) adopts a similar smooth ambi-

guity approach, but focuses primarily on illustrative examples and does not provide formal results on equilibrium existence or comparative statics. [Auster and Kellner \(2022\)](#) study Dutch auctions under maxmin preferences, where information revelation can change the locally worst prior and thereby break the usual equivalence with the first-price auction.

Regarding bidders' preferences over auction formats, [Matthews \(1987\)](#) compares risk-averse buyers' interim preferences over first- and second-price auctions, while [Lo \(1998\)](#), in the maxmin framework, finds that the second-price auction is weakly preferred in symmetric equilibrium. [Song \(2025\)](#) study auctions with ambiguity-averse bidders as robust auctions with affiliated values.

Our paper contributes to this literature by providing an equilibrium analysis of first-price auctions under smooth ambiguity preferences, including existence, uniqueness, comparative statics, and implications for revenue and auction-format comparisons.

The remainder of the paper is organized as follows. [Section 2](#) presents the environment, characterizes the unique symmetric equilibrium, and studies the case of constant relative ambiguity aversion in detail. [Section 3](#) contains the comparative statics analysis. [Section 4](#) compares the auction formats in terms of revenue and bidders' interim preferences. [Section 5](#) concludes. Proofs are collected in the appendices.

2 First-Price Auctions under Ambiguity

We consider a first-price sealed-bid auction (FPA) with $N \geq 2$ bidders competing for a single indivisible good. Set $\mathcal{I} := \{1, \dots, N\}$. Each bidder privately observes their own valuation v_i in $V = [\underline{v}, \bar{v}] \subset [0, \infty)$, where $0 \leq \underline{v} < \bar{v} < \infty$. The interior of V is $V^\circ = (\underline{v}, \bar{v})$. The reserve price is normalized to zero.

Bidders face ambiguity about opponents' valuations, modeled by a nondegenerate (non-singleton) family $\mathcal{F}$ of candidate beliefs $\mathcal{F} := \{F_\theta : \theta \in \Theta\}$, indexed by a parameter set $\Theta = [\underline{\theta}, \bar{\theta}] \subset \mathbb{R}$. Each F_θ is a cumulative distribution function (CDF) with common support V , admitting a strictly positive density $f_\theta(v) := \partial_v F_\theta(v)$ on V . We assume that $(\theta, v) \mapsto F_\theta(v)$ and $(\theta, v) \mapsto f_\theta(v)$ are jointly continuous on $\Theta \times V$. For each model parameter $\theta \in \Theta$, the bidder considers others' valuations as independent with identical marginal distribution F_θ . We write $f_{\theta, -i}(v_{-i}) = \prod_{j \neq i} f_\theta(v_j)$.

We adopt the smooth ambiguity model introduced by [Klibanoff et al. \(2005\)](#). In this framework, a bidder evaluates a bid through a two-stage process. First, for each model parameter θ , the bidder computes expected utility using the corresponding candidate belief F_θ . Second, these first-stage expected utilities are aggregated using the fully supported second-order belief μ over the parameter set Θ and an ambiguity index ϕ that reflects the bidder's attitude toward ambiguity. The function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is assumed

to be continuous, strictly increasing, concave, and thrice continuously differentiable, with $0 < \phi'(0) < \infty$.¹

The bidders simultaneously submit sealed bids $b_1, \dots, b_N \in \mathbb{R}_+$. In the first-price auction, the highest bidder wins the object and pays her bid. Given the submitted bids, bidder i 's ex-post payoff is

$$u_i(v_i, b_1, \dots, b_N) = \begin{cases} v_i - b_i, & \text{if } b_i > \max_{j \neq i} b_j, \\ 0, & \text{if } b_i < \max_{j \neq i} b_j. \end{cases}$$

If there is a tie for the highest bid, ties are broken uniformly at random.

Any bid $b_i > v_i$ suffers from the winner's curse and it is weakly dominated by bidding v_i . We therefore restrict bidder i 's strategies to be bounded by their own valuation. A strategy for bidder i is thus a function $\beta_i : V \rightarrow \mathbb{R}_+$, with $0 \leq \beta_i(v_i) \leq v_i$. For bidder i , write

$$v_{-i} := (v_j)_{j \neq i}, \quad \beta_{-i}(v_{-i}) := (\beta_j(v_j))_{j \neq i}.$$

Bidder i 's *interim utility* from placing a bid b_i , given valuation v_i and opponents' strategies β_{-i} , is

$$U_i(b_i, \beta_{-i}, v_i) = \int_{\Theta} \phi \left(\int_{V^{N-1}} u_i(v_i, b_i, \beta_{-i}(v_{-i})) f_{\theta, -i}(v_{-i}) dv_{-i} \right) \mu(d\theta).$$

2.1 Equilibrium

We focus on symmetric interim equilibria of our game with non-Bayesian preferences. A strategy profile $(\beta_i)_{i \in \mathcal{I}}$ is an *interim equilibrium* if, for every $i \in \mathcal{I}$ and $v_i \in V$,

$$\beta_i(v_i) \in \arg \max_{b_i \in [0, v_i]} U_i(b_i, \beta_{-i}, v_i).$$

It is *symmetric* if there exists a function β such that $\beta_i = \beta$ for every $i \in \mathcal{I}$. A symmetric equilibrium is *non-decreasing* if its common function β is non-decreasing in own valuation. The interim equilibrium notion is natural in the auction context because bidders choose after observing their own valuations; the behaviorally relevant requirement is therefore optimality at every realized valuation.

We next motivate two natural assumptions in the auction context. As is well known, the reverse hazard rate plays an important role in auctions. Our first assumption

¹Lower boundedness and thrice continuous differentiability are commonly assumed, e.g., by Maskin and Riley (1984).

requires that there are worst and best beliefs according to the ordering induced by the reverse hazard rate. Let F and G be two cumulative distribution functions on V . We say that F dominates G in the *Reverse Hazard Rate (RHR) order*, denoted $F \succsim_{\text{RHR}} G$, if and only if $\frac{f(x)}{F(x)} \geq \frac{g(x)}{G(x)}$ for all $x \in V^\circ$.

Assumption 1 (RHR-ordered beliefs). For every $\theta' \geq \theta$ we have

$$\frac{f_{\theta'}(v)}{F_{\theta'}(v)} \geq \frac{f_\theta(v)}{F_\theta(v)} \quad \forall v \in V^\circ.$$

Define the most and least competitive endpoint distributions by $F^+ := F_{\bar{\theta}}$ and $F^- := F_{\underline{\theta}}$, respectively.

The RHR order has a direct interpretation in terms of competitive intensity. Conditional on a rival's valuation being below v , a larger $f_\theta(v)/F_\theta(v)$ assigns more local probability near v ; with $N - 1$ independent rivals, $(N - 1)f_\theta(v)/F_\theta(v)$ is the proportional rate at which the probability of beating all rivals rises with the cutoff. In the standard SEU benchmark, a higher RHR therefore yields higher equilibrium bids at every valuation. **Assumption 1** ranks the candidate beliefs by the competitive intensity of the corresponding auction environment, so ambiguity about opponents' valuation distribution can be viewed as uncertainty about which auction environment the bidder faces. Familiar RHR-ordered distribution families are collected in **Appendix E**.

An advantage of the smooth ambiguity model is that ambiguity attitudes admit Arrow–Pratt-type local indices based on the curvature of ϕ ; see **Klibanoff et al. (2005)** and **Fabbri, Principi, and Stanca (2024)**. Our second assumption concerns this measure of ambiguity aversion.

Assumption 2 (Ambiguity Attitude Regularity). Define relative ambiguity aversion by $\rho(x) := -x\phi''(x)/\phi'(x)$.

Either

- (i) ρ is weakly increasing, i.e., preferences exhibit weakly increasing relative ambiguity aversion;² or
- (ii) the ambiguity attitude has bounded oscillation, i.e., $\text{osc}(\rho) := \sup_x \rho(x) - \inf_x \rho(x) < 1$.

We maintain **Assumption 1** and one of the two alternatives in **Assumption 2** from now on.

Let us now come to our main theorem. For each model parameter θ , the highest rival has CDF $H_\theta(v) := F_\theta(v)^{N-1}$ and density $h_\theta(v) := (N - 1)F_\theta(v)^{N-2}f_\theta(v)$. When

²**Baillon and Placido (2019)** provide experimental evidence on monotonic ambiguity attitudes.

all opponents use a strictly increasing symmetric strategy β , a bid $\beta(x)$ gives expected payoff $H_\theta(x)[v - \beta(x)]$ under the candidate belief F_θ . This term will thus play a role in the ordinary differential equation that the symmetric equilibrium bidding strategy satisfies.

Theorem 1. *1. There exists a unique non-decreasing symmetric equilibrium in which the bidding strategy β solves the ordinary differential equation (ODE)*

$$\beta'(v) = \frac{\int_{\Theta} \phi'(H_\theta(v)(v - \beta(v))) h_\theta(v) \mu(d\theta)}{\int_{\Theta} \phi'(H_\theta(v)(v - \beta(v))) H_\theta(v) \mu(d\theta)} (v - \beta(v)), \quad v \in V^\circ, \quad \beta(\underline{v}) = \underline{v}. \quad (1)$$

2. Every non-decreasing symmetric equilibrium strategy β exhibits bid shading on the interior, i.e. $\beta(v) < v$ for $v \in V^\circ$, is strictly increasing and continuous, and satisfies the ODE (1).

Proof. See [Appendix A](#). □

Let us compare the equilibrium bid under ambiguity with the bidding behavior of a Bayesian agent. Without ambiguity, there is a single known valuation distribution F ; write $H := F^{N-1}$ and $h := H'$. The standard Bayesian equilibrium bid satisfies the ordinary differential equation

$$\beta'(v) = \frac{h(v)}{H(v)} (v - \beta(v)) = (N-1) \frac{f(v)}{F(v)} (v - \beta(v)).$$

Smooth ambiguity preserves this form but weighs the densities h_θ and the distribution functions H_θ with the respective marginal ambiguity-adjusted expected utility term $\phi'(H_\theta(v)(v - \beta(v)))$. In Bayesian first-price auction, the reverse hazard rate $\frac{h(v)}{H(v)}$ determines how aggressive the bidding is. Under Knightian uncertainty, the corresponding term is an endogenous weighted average of candidate highest-rival reverse hazard rates, with weights determined by marginal ambiguity utility.

In the ambiguity-neutral case, one would conjecture that the equilibrium bid coincides with the equilibrium bid of a Bayesian agent who uses the predictive (average) prior $\bar{F}(v) = \int_{\Theta} F_\theta(v) \mu(d\theta)$. This is true for $N = 2$ bidders, but interestingly wrong for more than two bidders as the next corollary explains.

Corollary 1 (Ambiguity neutrality). *Consider the ambiguity-neutral case $\phi(x) = x$. Define*

$$F_{N-1}(v) := \left(\int_{\Theta} F_\theta(v)^{N-1} \mu(d\theta) \right)^{1/(N-1)}, \quad f_{N-1}(v) := F'_{N-1}(v).$$

Then the equilibrium bidding function under ambiguity is equal to that of a standard Bayesian first-price auction with independent private values (IPV) distributed according to the prior F_{N-1} .

For $N > 2$, F_{N-1} differs from the predictive prior $\bar{F}$ because $\int_{\Theta} F_{\theta}(v)^{N-1} \mu(d\theta) \neq (\int_{\Theta} F_{\theta}(v) \mu(d\theta))^{N-1}$ in general.

Proof. See [Appendix A](#). □

We next show that the bidding function under the smooth ambiguity model lies between the bidding functions of Bayesian agents under the least and most competitive endpoint distributions, respectively.

Proposition 1 (Comparison with Endpoint Distributions). *Denote by β^+ resp. β^- the equilibrium bid of a Bayesian agent under the most competitive endpoint distribution F^+ resp. least competitive endpoint distribution F^- according to Assumption 1. We have*

$$\beta^-(v) \leq \beta(v) \leq \beta^+(v)$$

for all $v \in V$.

Proof. See [Appendix B](#). □

In the next step, we compare our results to the benchmark case of a pessimistic (maxmin) ambiguity-averse agent in the sense of [Gilboa and Schmeidler \(1989\)](#) with preferences

$$U_i^{\min}(b_i, \beta_{-i}, v_i) = \min_{\theta \in \Theta} \int_{V^{N-1}} u_i(v_i, b_i, \beta_{-i}(v_{-i})) f_{\theta, -i}(v_{-i}) dv_{-i}.$$

Under [Assumption 1](#), the pointwise lower envelope of the candidate CDFs is F^+ . [Lo \(1998\)](#) has shown that $\beta^{\min} = \beta^+$; an extremely pessimistic agent assumes the most competitive endpoint distribution. [Proposition 1](#) immediately implies

$$\beta(v) \leq \beta^{\min}(v) \quad \text{for every } v \in V.$$

Thus the smooth-ambiguity equilibrium bid is bounded above by Lo's maxmin bidding benchmark.

2.2 Constant Relative Ambiguity Aversion

In this section, we treat the particularly transparent case of constant relative ambiguity aversion (CRAA), i.e.

$$\phi(x) = \begin{cases} \frac{x^{1-a} - 1}{1-a}, & a \geq 0, a \neq 1, \\ \ln x, & a = 1. \end{cases}$$

For $a > 0$, CRAA indices fall outside the standing regularity conditions at zero: $\phi'(0) = \infty$ when $0 < a < 1$, while ϕ is not finite at zero when $a \geq 1$. We therefore analyze CRAA separately. It turns out that with CRAA, we can identify exactly the distorted ambiguity-adjusted distribution of a virtual Bayesian agent to which the equilibrium bid under ambiguity corresponds.

Proposition 2. *Suppose bidders have CRAA preferences with ambiguity $a > 0$. Define the ambiguity-adjusted distribution by*

$$F_a(v) := \begin{cases} \left(\int_{\Theta} F_{\theta}(v)^{(N-1)(1-a)} \mu(d\theta) \right)^{1/[(N-1)(1-a)]}, & a \neq 1, \\ \exp \left(\int_{\Theta} \log F_{\theta}(v) \mu(d\theta) \right), & a = 1, \end{cases} \quad v \in V.$$

The displayed formula is understood on V° , with $F_a(\underline{v}) = 0$ defined by continuous extension. Then the equilibrium bid under ambiguity β coincides with the equilibrium bid of Bayesian bidders with prior F_a .

In particular, the equilibrium bid is explicitly given by

$$\beta(v) = v - \frac{\int_{\underline{v}}^v F_a(t)^{N-1} dt}{F_a(v)^{N-1}}. \quad (2)$$

Proof. See [Appendix C.2](#). □

Let us illustrate the previous result in a common numerical environment. Take

$$V = [0, 1], \quad N = 4, \quad k \sim \text{Unif}[1, 2], \quad F_k(v) = v^k.$$

For this family,

$$f_k(v) = kv^{k-1}, \quad H_k(v) = v^{3k}, \quad h_k(v) = 3kv^{3k-1}.^3$$

³Although the power family of distributions does not satisfy the assumption $f_k(0) > 0$, all our results hold for the family.

Moreover, for $v > 0$,

$$\frac{f_k(v)}{F_k(v)} = \frac{k}{v}.$$

Thus a larger value of the model parameter k indexes a more competitive distribution, with $F^- = F_1$ and $F^+ = F_2$. The singleton-belief FPA benchmarks are

$$\beta_1(v) = \frac{3}{4}v, \quad \beta_2(v) = \frac{6}{7}v.$$

For a CRAA coefficient a , let $r := 3(1 - a)$. If $a \neq 1$ and $v \in (0, 1)$, the effective prior in [Proposition 2](#) is

$$F_a(v) = \left[\int_1^2 v^{rk} dk \right]^{1/r} = \left[\frac{v^{2r} - v^r}{r \log v} \right]^{1/r}.$$

For $a = 1$,

$$F_{a=1}(v) = \exp \left(\int_1^2 k \log v dk \right) = v^{3/2}.$$

The corresponding equilibrium bid is

$$\beta_a(v) = v - \frac{\int_0^v F_a(t)^3 dt}{F_a(v)^3},$$

and, in particular, $\beta_{a=1}(v) = \frac{9}{11}v$. [Fig. 1](#) displays representative equilibrium bids for $a \in \{0.2, 1, 3\}$ together with the two singleton-belief benchmarks. Its right panel reports bid differences relative to the logarithmic case $a = 1$, making the ordering among the closely spaced CRAA bids visible.

3 Comparative Statics

In this section, we investigate the comparative statics of ambiguity aversion and the number of bidders. Ambiguity aversion leads to more aggressive bidding, as intuition suggests. In contrast to the Bayesian case, however, a larger number of bidders need not lead to more aggressive bidding, even under CRAA. This is a sharp contrast between Knightian uncertainty and the Bayesian case.

Ambiguity Aversion: between neutrality and maxmin. We first investigate how differences in ambiguity aversion translate into equilibrium bidding behavior. Let preferences A and B differ only in their ambiguity indices ϕ_A and ϕ_B . Preference A is more ambiguity averse than B if $\phi_A = \chi \circ \phi_B$ for some strictly increasing and concave

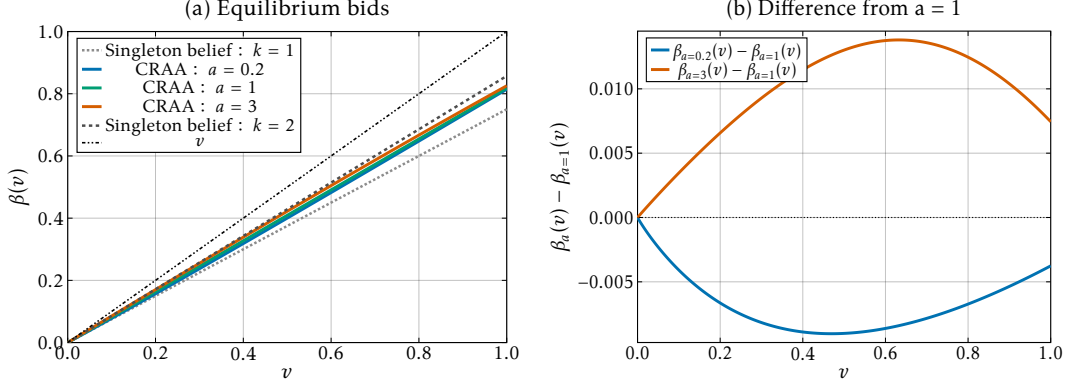


Figure 1: CRAA equilibrium bids in the power-family illustration. The left panel reports bid levels, while the right panel reports differences relative to $a = 1$. Equilibrium bids increase with ambiguity aversion and remain between the singleton-belief benchmarks $k = 1$ and $k = 2$. Here $F_k(v) = v^k$, $k \sim \text{Unif}[1, 2]$, and $N = 4$.

function χ .

Theorem 2. *Higher ambiguity aversion leads to more aggressive bidding.*

Formally, let β_A and β_B denote the symmetric, non-decreasing equilibrium bidding functions of bidders with ambiguity indices ϕ_A and ϕ_B , respectively. Assume that A is more ambiguity averse than B in the sense that $\phi_A = \chi \circ \phi_B$ for some strictly increasing and concave function χ . Then, for every $v \in V$,

$$\beta_A(v) \geq \beta_B(v).$$

Proof. See [Appendix B](#). □

[Raymond and Song \(2026\)](#) develop a general framework for monotone comparative statics under non-expected utility. In the smooth-ambiguity application closest to ours, their sufficient condition requires the density $f_\theta(v)$ to be log-supermodular in (v, θ) . Our [Assumption 1](#) is weaker: the RHR order imposed there is equivalent to the cumulative distribution function $F_\theta(v)$ being log-supermodular in (v, θ) . The ambiguity weights in [Section 2.1](#) make the mechanism precise. Greater ambiguity aversion shifts the endogenous weights toward model parameters with lower winning probabilities and, under the RHR order, higher reverse hazard rates. The ambiguity-adjusted coefficient therefore rises, and bidders reduce bid shading.

We now study the limiting behavior of bidding functions as ambiguity aversion approaches infinity. As intuition suggests, equilibrium bids tend then to the equilibrium bids under maxmin expected utility.

Theorem 3 (Extreme ambiguity aversion). *Let $\{\phi_k\}_{k \geq 1}$ be a sequence of ambiguity indices satisfying our standing assumptions. For $\alpha_k(x) := -\frac{\phi_k''(x)}{\phi_k'(x)}$, suppose $\alpha_{k+1}(x) \geq \alpha_k(x)$ for every $x \in [0, \bar{v}]$ and every k , and $\inf_{x \in [0, \bar{v}]} \alpha_k(x) \rightarrow +\infty$. Let β_k denote the unique continuous non-decreasing symmetric equilibrium bidding function associated with ϕ_k .*

Then

$$\beta_k(v) \uparrow \beta^{\min}(v) \quad \text{for every } v \in V.$$

Proof. See [Appendix B](#). □

The CRAA case admits a parallel direct limiting argument through its ambiguity-adjusted prior representation.

Remark 1. *For CRAA preferences, the effective-prior representation yields a transparent direct limit. The function $F_a(v)$ is a generalized mean of the candidate CDFs $\{F_\theta(v)\}_{\theta \in \Theta}$. By the generalized-mean limit,*

$$\lim_{a \rightarrow \infty} F_a(v) = \min_{\theta \in \Theta} F_\theta(v) = F_{\bar{\theta}}(v) \quad \forall v \in V.$$

Writing β_a for the equilibrium bid associated with parameter a , the formula in [Eq. \(2\)](#), together with bounded convergence, therefore gives $\beta_a(v) \rightarrow \beta^{\min}(v)$ pointwise.

Changing the number of bidders. There is more competition when the number of bidders increases. Thus, one expects more aggressive bidding in this case. In the Bayesian case, this holds true. For a single known distribution F , the equilibrium bid $\beta_N^F(v)$ satisfies

$$\begin{aligned} v - \beta_N^F(v) &= \frac{\int_{\underline{v}}^v F(t)^{N-1} dt}{F(v)^{N-1}} \\ &= \int_{\underline{v}}^v \exp \left\{ -(N-1) \int_t^v \frac{f(u)}{F(u)} du \right\} dt. \end{aligned}$$

This expression decreases with N , and hence $\beta_N^F(v)$ increases. Note that the exponent contains the integral of the reverse hazard rate $r = \frac{f}{F}$ from t to v , with a negative sign; it is not simply $N-1$ times the reverse hazard rate r .

Under ambiguity neutrality, however, from [Corollary 1](#),

$$\beta_N^0(v) = v - \frac{\int_{\underline{v}}^v \int_{\Theta} F_\theta(t)^{N-1} \mu(d\theta) dt}{\int_{\Theta} F_\theta(v)^{N-1} \mu(d\theta)}.$$

Due to the average over the second-order belief μ , it is no longer possible to write this expression in terms of the reverse hazard rate. Increasing N can now have two opposing effects. Conditional on a fixed model parameter θ , more rivals reduce bid shading. At the same time, the factors $F_\theta(v)^{N-1}$ change the relative weights assigned to candidate beliefs. Under the RHR order, increasing N shifts relative weight toward candidate beliefs with larger CDFs and lower reverse hazard rates, which can increase bid shading. Neither effect dominates in general.

We illustrate this issue in the case of constant relative ambiguity aversion.

Example 1. Let $V = [0, 1]$, let the model parameter satisfy $k \in [0, 1]$, and set

$$F_k(v) = v \exp\{-10k(1-v)\}.$$

Let the second-order belief have density

$$\frac{d\mu}{dk}(k) = 0.1(1-k)^9 + 9.9k^9, \quad k \in [0, 1].$$

This density is strictly positive and integrates to one; equivalently, it is the mixture of two Beta-distributions, $\mu = 0.01 \text{Beta}(1, 10) + 0.99 \text{Beta}(10, 1)$. Moreover,

$$f_k(v) = e^{-10k(1-v)}(1 + 10kv) > 0 \quad \text{for every } (k, v) \in [0, 1]^2,$$

and, for $v > 0$,

$$\frac{f_k(v)}{F_k(v)} = \frac{1}{v} + 10k,$$

so the model family is strictly RHR ordered.

Take CRAA coefficient $a = 1/2$, and let β_N denote the equilibrium bid when the total number of bidders is N , holding all other primitives fixed. By the effective-prior representation in [Proposition 2](#), the effective highest-rival CDF is

$$\widehat{H}_N(v) = \left[\int_0^1 F_k(v)^{(N-1)/2} \mu(dk) \right]^2 = v^{N-1} \left[\int_0^1 e^{-5(N-1)k(1-v)} \mu(dk) \right]^2,$$

and hence

$$\beta_N(v) = v - \frac{\int_0^v \widehat{H}_N(t) dt}{\widehat{H}_N(v)}.$$

At $v = 1/2$, the generated values are

$$\beta_3(1/2) \approx 0.429, \quad \beta_4(1/2) \approx 0.413, \quad \beta_5(1/2) \approx 0.417.$$

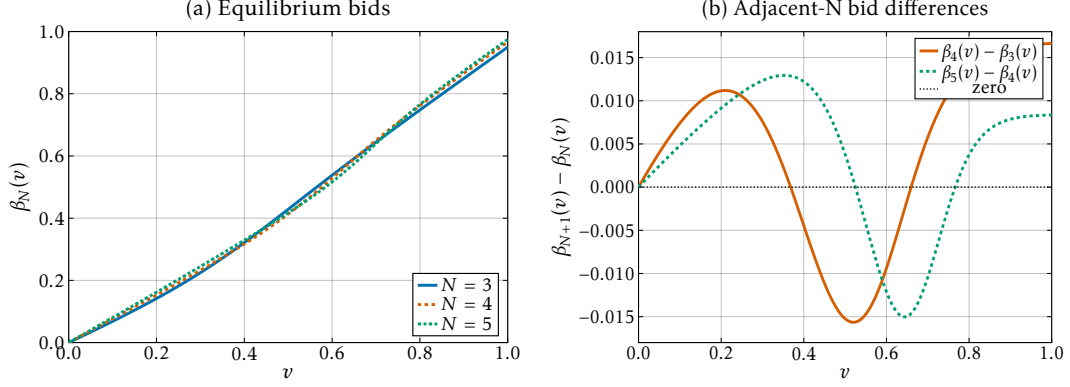


Figure 2: Comparative statics for increasing the number of bidders with constant relative ambiguity aversion. Panel (a) reports equilibrium bids for $N \in \{3, 4, 5\}$. Panel (b) reports adjacent- N bid differences; negative values show that adding a bidder can lower the equilibrium bid, while the changes of sign show that the comparison depends on valuation. Here $F_k(v) = ve^{-10k(1-v)}$, the second-order density is $0.1(1-k)^9 + 9.9k^9$, and $a = 1/2$.

Thus the equilibrium bid first falls and then rises as N increases from three to five.

Indeed, $\beta_4 - \beta_3$ is negative on the generated interval $(0.368, 0.660)$, while $\beta_5 - \beta_4$ is negative on $(0.525, 0.766)$. Thus the bid functions have no pointwise ordering in N .

4 Comparing Auction Formats

4.1 Revenue Comparison

We next compare the revenue of the first-price and the second-price auction from the seller's perspective. Recall that in a second-price auction, truth-telling remains the weakly dominant strategy even under ambiguity. We thus assume this equilibrium bidding behavior for the second-price auction.

In order to evaluate (ex-ante) revenue, we first have to decide how the seller deals with the ambiguity of the model. There are two natural approaches. If we assume that the second-order belief μ describes the objective model uncertainty of the environment that is common knowledge, then the seller naturally considers the expected revenue under the predictive prior by averaging model-specific expected revenue across model parameters. We call this approach the predictive criterion. In other environments, we might also think of an informed seller who knows or insists to use one specific model parameter $\theta \in \Theta$. We call this criterion the model-based criterion.

The predictive criterion. Let us describe how the expected revenue is computed by the seller under the predictive criterion. Conditional on a given model parameter θ ,

the N valuations are independently distributed according to F_θ . The associated joint law is thus

$$Q(d\theta, dv_1, \dots, dv_N) = \mu(d\theta) \prod_{i=1}^N dF_\theta(v_i), \quad (3)$$

We use Q as the seller's ex ante evaluation law and define the respective revenues of the first-price and second-price auction as

$$R_\phi^{\text{FPA}} := \int_{\Theta} \int_{V^N} \max_{1 \leq i \leq N} \beta_\phi(v_i) \prod_{i=1}^N dF_\theta(v_i) \mu(d\theta) = \int_{\Theta} \int_V \beta_\phi(v) dF_\theta(v)^N \mu(d\theta), \quad (4)$$

$$R^{\text{SPA}} := \int_{\Theta} \mathbb{E}^\theta[v_{(2)}] \mu(d\theta) = \int_{\Theta} \int_{V^N} v_{(2)} \prod_{i=1}^N dF_\theta(v_i) \mu(d\theta), \quad (5)$$

where the second equality in [Eq. \(4\)](#) uses monotonicity of the equilibrium bid β_ϕ and the fact that the highest valuation has cumulative distribution $F_\theta(v)^N$ under model parameter θ . For the second-price auction, $v_{(2)}$ denotes the second-highest valuation among bidders.

For each $F \in \mathcal{F}$, let $\beta_{\{F\}}$ denote the symmetric equilibrium bidding function in the standard risk-neutral FPA with singleton prior F . For each fixed θ , standard revenue equivalence gives

$$\mathbb{E}^\theta[v_{(2)}] = \int_V \beta_{\{F_\theta\}}(v) d(F_\theta(v)^N).$$

Integrating over θ , we obtain

$$R^{\text{SPA}} = \int_{\Theta} \int_V \beta_{\{F_\theta\}}(v) d(F_\theta(v)^N) \mu(d\theta). \quad (6)$$

In the first step, we show that the seller prefers the second-price auction when bidders are ambiguity-neutral.

Proposition 3 (Predictive revenue under ambiguity neutrality). *With ambiguity-neutral bidders, the second-price auction generates higher expected revenue under the predictive prior for the seller compared to the first-price auction.*

Proof. See [Appendix D](#). □

With ambiguity-neutral bidders, expected revenue under the first-price auction is lower than expected revenue under the second-price auction. This is due to the effect that Jensen's inequality gives

$$\int_{\Theta} F_\theta(v)^N \mu(d\theta) \geq \bar{F}(v)^N,$$

with strict inequality whenever $\theta \mapsto F_\theta(v)$ is not μ -almost surely constant. The seller thus works with a less competitive distribution for evaluation bids than the one given by N independent draws from the predictive prior. From that perspective, expected bids are lower than the expected bids in a first-price auction where bidders' valuations are drawn independently from the predictive prior, leading to lower expected revenue.

On the other hand, we know from [Lo \(1998\)](#) that the first-price auction yields higher revenue than the second-price auction when bidders are ambiguity-averse in the maxmin sense. This conclusion is also in line with our findings. [Theorem 2](#) shows that bidders bid more aggressively when ambiguity aversion increases, and their bids converge to the maxmin bid as ambiguity aversion tends to infinity. We now show that the predictive revenue of the first-price auction converges monotonically to a higher value than the predictive revenue of the second-price auction.

Proposition 4 (Predictive revenue under large ambiguity aversion). *For sufficiently large ambiguity aversion, the FPA generates strictly higher predictive revenue than the SPA.*

More formally: Let $\{\phi_\ell\}_{\ell \geq 1}$ be a sequence of ambiguity indices satisfying the hypotheses of [Theorem 3](#), and let β_ℓ denote the corresponding FPA equilibrium bids. Then the expected revenue in the first-price auction is monotonically increasing and there exists $\ell_0 < \infty$ such that $R_{\phi_\ell}^{\text{FPA}} > R^{\text{SPA}}$ for every $\ell \geq \ell_0$.

Proof. See [Appendix D](#). □

Example 2. Consider again the power-family environment from [Section 2.2](#):

$$V = [0, 1], \quad N = 4, \quad k \sim \text{Unif}[1, 2], \quad F_k(v) = v^k.$$

For each CRAA coefficient $a \geq 0$, let β_a denote the symmetric equilibrium bid. For $a > 0$, it is computed from the effective-prior representation in [Proposition 2](#); at $a = 0$, we use its ambiguity-neutral specialization. Under the seller's common-parameter predictive law, one model parameter k is drawn for the auction and, conditional on k , all four valuations are drawn independently from F_k .

Since the maximal valuation has the density

$$4F_k(v)^3 f_k(v) = 4kv^{4k-1}$$

under model parameter k , the predictive revenue of the first-price auction is

$$R_{\phi_a}^{\text{FPA}} = \int_1^2 \int_0^1 \beta_a(v) 4kv^{4k-1} dv dk.$$

Truthful bidding is dominant in the SPA. We apply classical Bayesian revenue equivalence separately for each fixed model parameter k : the revenue of the second-price auction under prior F_k is equal to the revenue of the first-price auction under the same prior, but with the corresponding Bayesian equilibrium bid $\beta_{\{F_k\}}$, not with the smooth-ambiguity bid β_a . Because

$$\beta_{\{F_k\}}(v) = \frac{3k}{3k+1}v,$$

we obtain

$$\begin{aligned} R^{\text{SPA}} &= \int_1^2 \int_0^1 \beta_{\{F_k\}}(v) 4kv^{4k-1} dv dk \\ &= 0.695. \end{aligned}$$

The most competitive model parameter is $k = 2$, so the most competitive endpoint distribution, and hence the worst case for a maxmin agent, is F_2 , resulting in an equilibrium bid $\beta^{\min} = \beta_{\{F_2\}}$. Its predictive-revenue benchmark is

$$R_{\infty}^{\text{FPA}} := \int_1^2 \int_0^1 \beta^{\min}(v) 4kv^{4k-1} dv dk = 0.731.$$

At ambiguity neutrality, predictive FPA revenue is 0.691. It is weakly increasing, consistently with [Theorem 2](#). It eventually exceeds predictive SPA revenue and moves toward the MEU endpoint, illustrating the conclusion of [Proposition 4](#); for CRAA, the limiting step follows from the direct effective-prior argument after [Theorem 3](#). The computed revenue curves cross once, at $a^* = 0.544$. At the largest displayed coefficient $a = 20$, the distance from predictive FPA revenue to the MEU benchmark is 0.006.

All computed bids are non-decreasing in valuation.

The model-based criterion. Let us now consider the case when the seller knows the true model parameter θ and evaluates revenue under the corresponding distribution F_{θ} . Equivalently, we evaluate revenue separately under each candidate belief $F \in \mathcal{F}$. Although the ranking of the two formats may vary across candidate beliefs, the range of FPA revenues is contained in the corresponding range of SPA revenues.

For each $F \in \mathcal{F}$, define the FPA revenue generated by the ambiguity-equilibrium

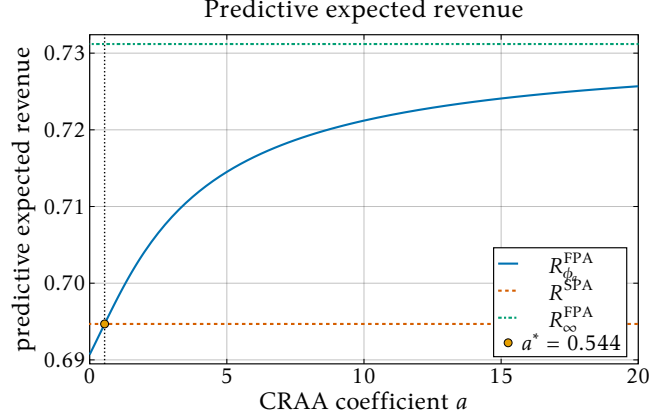


Figure 3: Predictive seller revenue under CRAA in the power-family illustration. The figure compares predictive FPA revenue with predictive SPA revenue as the CRAA coefficient increases; the upper horizontal line is the MEU endpoint benchmark R_{∞}^{FPA} . The computed FPA revenue eventually exceeds SPA revenue. Here $F_k(v) = v^k$, $k \sim \text{Unif}[1, 2]$, and $N = 4$. The unique computed crossing occurs at $a^* = 0.544$.

bid β and the SPA revenue, expressed using conditional revenue equivalence, by

$$R^{\text{FPA}}(F) := \int_V \beta(v) d(F(v)^N),$$

$$R^{\text{SPA}}(F) := \int_V \beta_{\{F\}}(v) d(F(v)^N),$$

where $\beta_{\{F\}}$ is the symmetric equilibrium bid in the standard risk-neutral FPA under the singleton prior F . The dependence of $R^{\text{FPA}}(F)$ on the maintained ambiguity index is suppressed in the notation.

Proposition 5 (Model-based revenue comparison). *For any $\theta', \theta \in \Theta$ with $\theta' \geq \theta$,*

$$R^{\text{FPA}}(F_{\theta'}) \geq R^{\text{FPA}}(F_{\theta}) \quad \text{and} \quad R^{\text{SPA}}(F_{\theta'}) \geq R^{\text{SPA}}(F_{\theta}).$$

The inequalities are strict whenever $F_{\theta'} \neq F_{\theta}$. Moreover,

$$R^{\text{FPA}}(F^-) > R^{\text{SPA}}(F^-), \quad R^{\text{FPA}}(F^+) < R^{\text{SPA}}(F^+).$$

That is, an informed seller prefers the first-price auction for less competitive models and the second-price auction for more competitive models.

Consequently, the two formats generate the same revenue for at least one

$$\theta \in (\underline{\theta}, \bar{\theta}).$$

Proof. See [Appendix D](#). □

Define the lower and upper revenue envelopes over $\mathcal{F}$ by

$$\begin{aligned}\underline{R}^{\text{FPA}} &:= \inf_{F \in \mathcal{F}} R^{\text{FPA}}(F), & \overline{R}^{\text{FPA}} &:= \sup_{F \in \mathcal{F}} R^{\text{FPA}}(F), \\ \underline{R}^{\text{SPA}} &:= \inf_{F \in \mathcal{F}} R^{\text{SPA}}(F), & \overline{R}^{\text{SPA}} &:= \sup_{F \in \mathcal{F}} R^{\text{SPA}}(F).\end{aligned}$$

The endpoint comparison uses the bid bounds in [Proposition 1](#).

Corollary 2 (Revenue interval compression). *The interval of possible revenues across candidate beliefs $F \in \mathcal{F}$ is smaller under the first-price auction compared to the second-price auction:*

$$\underline{R}^{\text{FPA}} \geq \underline{R}^{\text{SPA}}, \quad \overline{R}^{\text{FPA}} \leq \overline{R}^{\text{SPA}}.$$

Proof. See [Appendix D](#). □

Example 3. Consider again the power-family environment from [Section 2.2](#), where $V = [0, 1]$, $N = 4$, $k \sim \text{Unif}[1, 2]$, and $F_k(v) = v^k$. Fix the ambiguity aversion at $a = 1$. From the seller's perspective, higher parameter values k are better as they lead to more competitive distributions of bidder valuations. This example confirms [Proposition 5](#): the seller prefers the FPA for less competitive models and the SPA for more competitive models.

As shown above, the effective prior and equilibrium bid are

$$F_{a=1}(v) = v^{3/2}, \quad \beta_{a=1}(v) = \frac{9}{11}v.$$

Under model parameter k , the seller's expected revenue is

$$R_{a=1}^{\text{FPA}}(k) = \int_0^1 \beta_{a=1}(v) 4k v^{4k-1} dv = \frac{9}{11} \frac{4k}{4k+1}.$$

Under model parameter k , the CDF of the second-highest valuation is

$$P^k(v_{(2)} \leq v) = F_k(v)^4 + 4[1 - F_k(v)]F_k(v)^3 = 4v^{3k} - 3v^{4k}.$$

Using classical revenue equivalence, we have

$$R^{\text{SPA}}(k) = \mathbb{E}^k[v_{(2)}] = \int_0^1 [1 - 4v^{3k} + 3v^{4k}] dv = \frac{3k}{3k+1} \frac{4k}{4k+1}.$$

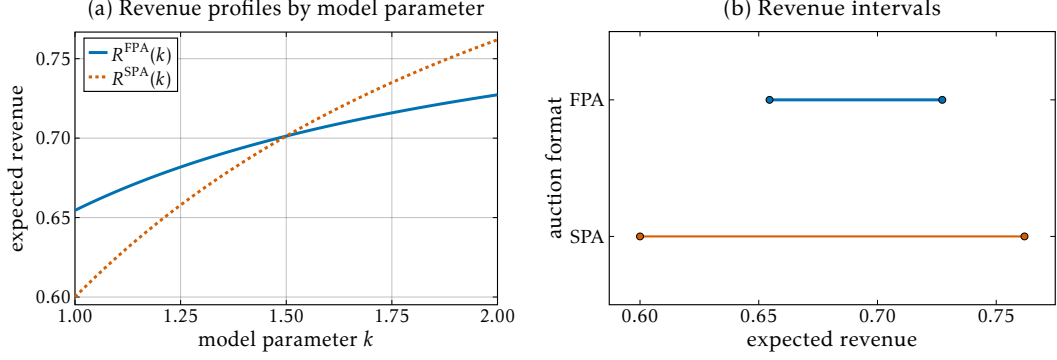


Figure 4: CRAA revenue-interval compression in the power-family illustration. Panel (a) compares FPA and SPA revenue across the model parameter k , while panel (b) displays the corresponding revenue intervals. The FPA raises the lower endpoint and lowers the upper endpoint relative to the SPA. Here $F_k(v) = v^k$, $k \sim \text{Unif}[1, 2]$, $a = 1$, and $N = 4$.

Both revenue profiles are increasing in k . The resulting revenue intervals are

$$R_{a=1}^{\text{FPA}}([1, 2]) = [0.655, 0.727], \quad R^{\text{SPA}}([1, 2]) = [0.600, 0.762].$$

The seller prefers the first-price auction for $k < 1.5$, is indifferent at $k = 1.5$, and prefers the second-price auction for $k > 1.5$; see [Fig. 4](#).

4.2 Bidders' Interim Preferences

Following [Matthews \(1987\)](#), we compare the FPA and SPA from the bidders' point of view. Our criterion is a bidder's equilibrium interim utility at each realized valuation. In the standard quasilinear IPV auctions, payoff equivalence gives a bidder the same interim utility in the two formats at every valuation. This equivalence can fail even without ambiguity once utility over money is nonlinear. [Matthews \(1987\)](#) shows that, with independent private values, bidders with decreasing absolute risk aversion prefer the SPA, bidders with constant absolute risk aversion are indifferent, and bidders with increasing absolute risk aversion prefer the FPA.

[Lo \(1998\)](#) obtains another uniform ranking under maxmin expected utility: when the two formats use the same reserve price, every bidder weakly prefers the SPA at every valuation at which she participates. Smooth ambiguity, by contrast, does not generally yield a uniform ranking. The two formats generate different profiles of expected payoffs under the candidate beliefs, which are then aggregated nonlinearly; the preferred format may therefore depend on the ambiguity index, the family of candidate beliefs, the second-order belief, and the bidder's valuation. This motivates our focus on two classes of ambiguity attitudes for which the ranking is sharp. CRAA bidders weakly prefer

the SPA, whereas bidders with IAAA weakly prefer the FPA, with CAAA as a special case. These classes do not exhaust smooth ambiguity preferences: we also provide an example outside them in which the ranking crosses with valuation, so that bidders with different valuations prefer different auction formats.

For an auction format $\mathcal{A} \in \{\text{FPA}, \text{SPA}\}$, write $U_i^{\mathcal{A}}(v) := U_i^{\mathcal{A}}(\beta_i^{\mathcal{A}}(v), \beta_{-i}^{\mathcal{A}}, v)$. A bidder with valuation v weakly prefers format $\mathcal{A}$ to format $\mathcal{B}$ if $U_i^{\mathcal{A}}(v) \geq U_i^{\mathcal{B}}(v)$. We say that bidders weakly prefer $\mathcal{A}$ to $\mathcal{B}$ if this inequality holds for every $v \in V$.

Recall that absolute ambiguity aversion is $\alpha(x) := -\phi''(x)/\phi'(x)$. Preferences exhibit increasing absolute ambiguity aversion (IAAA) when α is weakly increasing; CAAA is the constant special case.

Proposition 6 (CRAA bidder preference). *Suppose bidders have CRAA preferences. Then bidders weakly prefer the SPA to the FPA at every valuation:*

$$U^{\text{SPA}}(v) \geq U^{\text{FPA}}(v) \quad \forall v \in V.$$

Proof. See [Appendix C.3](#). □

Proposition 7 (IAAA bidder preference). *Suppose the maintained equilibrium assumptions hold and bidders' smooth ambiguity preferences exhibit IAAA. Then bidders weakly prefer the FPA to the SPA at every valuation:*

$$U^{\text{FPA}}(v) \geq U^{\text{SPA}}(v) \quad \forall v \in V.$$

In particular, the result applies to CAAA.

Proof. See [Appendix C.3](#). □

The bidder-preference reversal arises because IAAA, including CAAA, and CRAA evaluate ambiguity over payoffs under the candidate beliefs in fundamentally different ways. Under IAAA, low payoffs under the candidate beliefs receive disproportionately high marginal ambiguity utility. This makes the bidder especially sensitive to the downside states in which competition is strong. The first-price auction mitigates this downside exposure because, conditional on winning, the bidder's surplus is fixed by her own bid rather than varying across candidate beliefs. In this sense, the first-price auction provides insurance across candidate beliefs, which is valued under IAAA. Under CRAA, however, the relevant force is not downside protection but aggregation. Constant relative ambiguity aversion yields a homothetic aggregator, so the second-price auction benefits from aggregating payoff components within each candidate belief before ambiguity is evaluated. This aggregation advantage dominates the insurance role

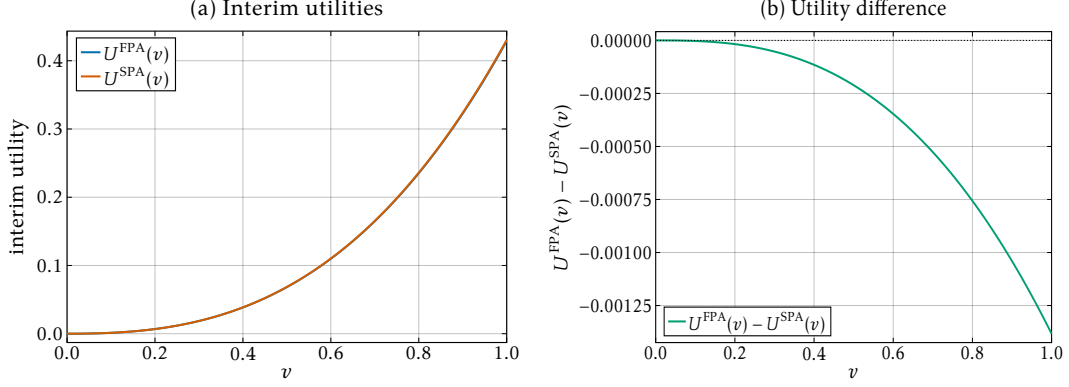


Figure 5: CRAA interim-utility comparison in the power-family illustration. The left panel reports $U^{\text{FPA}}(v)$ and $U^{\text{SPA}}(v)$, while the right panel reports their difference. Here $F_k(v) = v^k$, $k \sim \text{Unif}[1, 2]$, $N = 4$, and $\phi(x) = \sqrt{x}$.

of the first-price auction, leading CRAA bidders to prefer the second-price auction. Thus, the preference ranking reverses because IAAA emphasizes protection against unfavorable candidate beliefs, whereas CRAA emphasizes the relative aggregation of payoffs across candidate beliefs.

Example 4. *Continue with the power-family environment introduced above. For each model parameter k ,*

$$H_k(v) = v^{3k}, \quad A_k(v) := \int_0^v H_k(t) dt = \frac{v^{3k+1}}{3k+1}.$$

Given an FPA equilibrium bid β , define the payoffs under model parameter k :

$$X_k^{\text{FPA}}(v) = H_k(v)[v - \beta(v)], \quad X_k^{\text{SPA}}(v) = A_k(v).$$

The smooth-ambiguity interim utilities are

$$U^{\text{FPA}}(v) = \int_1^2 \phi(X_k^{\text{FPA}}(v)) dk, \quad U^{\text{SPA}}(v) = \int_1^2 \phi(X_k^{\text{SPA}}(v)) dk.$$

First let $\phi(x) = \sqrt{x}$. The exact CRAA effective-prior representation with $a = \frac{1}{2}$ yields the FPA equilibrium bid. The resulting interim-utility comparison is shown in Fig. 5; the SPA is weakly preferred at every valuation.

Next let

$$\phi(x) = \frac{1 - e^{-5x}}{5}.$$

The corresponding CAAA equilibrium yields the opposite ranking, as shown in Fig. 6;

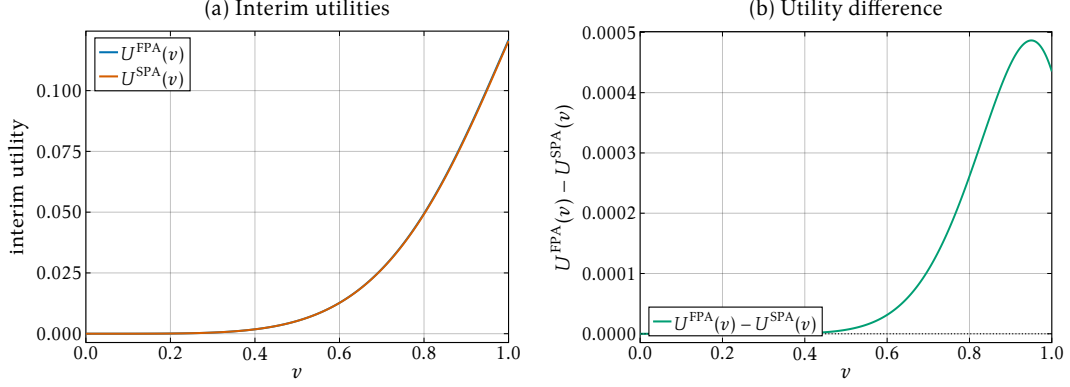


Figure 6: CAAA interim-utility comparison in the power-family illustration. The left panel reports $U^{\text{FPA}}(v)$ and $U^{\text{SPA}}(v)$, while the right panel reports their difference. The difference remains nonnegative but need not be monotone. Here $F_k(v) = v^k$, $k \sim \text{Unif}[1, 2]$, $N = 4$, and $\phi(x) = (1 - e^{-5x})/5$.

the FPA is weakly preferred at every valuation. This CAAA numerical example illustrates the more general IAAA result above.

The following example shows that the auction-format preference ranking can cross as the bidder's valuation varies.

Example 5. *Continue with the power-family environment*

$$V = [0, 1], \quad N = 4, \quad k \sim \text{Unif}[1, 2], \quad F_k(v) = v^k.$$

Let

$$\phi(z) = \int_0^z \exp \left\{ -9.01x + 30x^2 - \frac{100}{3}x^3 \right\} dx.$$

The utility comparison has one economically meaningful interior crossing at $v \approx 0.93956$. Bidders with valuations below the crossing prefer the FPA, whereas those with sufficiently high valuations prefer the SPA; see [Fig. 7](#).

5 Conclusion

This paper studies first-price sealed-bid auctions when bidders face ambiguity about the distribution of their opponents' valuations and evaluate such uncertainty through smooth ambiguity preferences. Our results show that this framework preserves enough structure to permit a sharp equilibrium analysis while generating implications that differ systematically from both the Bayesian and maxmin benchmarks. We establish existence and uniqueness of a symmetric interim equilibrium under suitable conditions

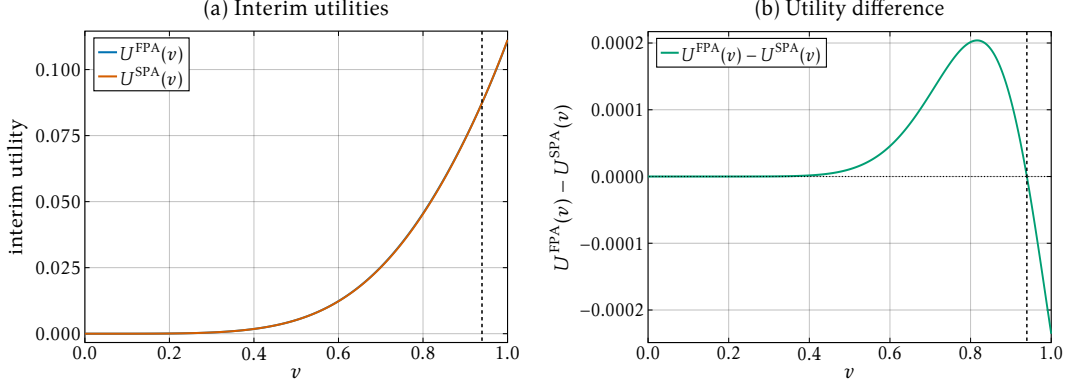


Figure 7: Valuation-dependent interim-utility comparison in the power-family illustration. The left panel reports $U^{\text{FPA}}(v)$ and $U^{\text{SPA}}(v)$, while the right panel reports their difference. The vertical line marks the crossing between FPA and SPA preferences under $F_k(v) = v^k$, $k \sim \text{Unif}[1, 2]$, and $N = 4$.

and characterize equilibrium bidding through a nonlinear differential equation in which ambiguity enters through an endogenous distortion of beliefs.

This characterization yields a transparent account of how ambiguity affects strategic bidding. Ambiguity aversion makes bidders more aggressive, and equilibrium bids remain pinned between the Bayesian bids associated with the least and most competitive candidate priors, converging to the maxmin benchmark as ambiguity aversion becomes extreme. At the same time, smooth ambiguity generates comparative statics that have no analogue in the standard model: in particular, the effect of additional competition need not be monotone, because the number of bidders changes not only the strategic environment but also the effective prior against which bidders optimize.

The analysis also has clear implications for auction design. From the seller's perspective, the revenue ranking between first-price and second-price auctions depends on how model uncertainty is evaluated. Under a predictive criterion, sufficiently strong ambiguity aversion can overturn the ambiguity-neutral ranking and make the first-price auction more profitable. Under a model-by-model criterion, the first-price auction performs better in less competitive environments and worse in more competitive ones, while compressing the range of possible revenues across models. From the bidders' perspective, smooth ambiguity does not imply a uniform ranking of auction formats, so ambiguity attitudes themselves become relevant for the comparison between standard auction mechanisms.

Taken together, these findings suggest that smooth ambiguity provides a useful middle ground for auction theory. It retains more behavioral richness than the Bayesian model, while remaining substantially more structured than worst-case approaches. As

a result, it offers a tractable way to study how model uncertainty shapes equilibrium behavior, revenue, and mechanism comparison in auctions.

Appendix

Appendix A Proof of the Equilibrium Characterization

The proof is organized into four lemmas. **Lemma 1** records the order implications of **Assumption 1** and gives the domination needed to differentiate under the second-order belief. **Lemma 2** proves necessity: in any non-decreasing equilibrium, bidders shade their bids at every valuation above $\underline{v}$; the bidding strategy has no pooling or jumps, is locally Lipschitz, and satisfies the ODE on the interior. **Lemma 3** proves existence by constructing the interior ODE path that can be continuously attached to the lower boundary. **Lemma 4** verifies optimality of the constructed solution and proves uniqueness.

Lemma 1 (Order bounds and differentiation under μ). *Suppose **Assumption 1** holds. Write $H^\pm := (F^\pm)^{N-1}$ and $h^\pm := (H^\pm)'$. For every $\theta \in \Theta$ and every $x \in V^\circ$,*

$$H^+(x) \leq H_\theta(x) \leq H^-(x), \quad \frac{h_{\bar{\theta}}(x)}{H_{\bar{\theta}}(x)} \geq \frac{h_\theta(x)}{H_\theta(x)} \geq \frac{h_{\underline{\theta}}(x)}{H_{\underline{\theta}}(x)}.$$

For every compact interval $I \subset V^\circ$, define

$$M_I := \max_{(\theta, x) \in \Theta \times I} h_\theta(x) < \infty, \quad m_I := \min_{(\theta, x) \in \Theta \times I} H_\theta(x) > 0.$$

Finally, define

$$\Psi(x, c) := \int_{\Theta} \phi(H_\theta(x)c) \mu(d\theta), \quad (x, c) \in V^\circ \times (0, \infty).$$

Then Ψ is continuously differentiable on compact subsets of $V^\circ \times (0, \infty)$, and

$$\begin{aligned} \Psi_x(x, c) &= \int_{\Theta} \phi'(H_\theta(x)c) h_\theta(x) c \mu(d\theta), \\ \Psi_c(x, c) &= \int_{\Theta} \phi'(H_\theta(x)c) H_\theta(x) \mu(d\theta). \end{aligned} \tag{7}$$

Moreover, $\Psi_c(x, c) > 0$.

Proof. The implication from the RHR order to first-order stochastic dominance is standard. With our convention, $F^+ \succeq_{\text{RHR}} F_\theta \succeq_{\text{RHR}} F^-$ implies $F^+(x) \leq F_\theta(x) \leq F^-(x)$; see **Shaked and Shanthikumar (2007)**. Raising these inequalities to the power $N - 1$

gives $H^+(x) \leq H_\theta(x) \leq H^-(x)$. Moreover,

$$\frac{h_\theta(x)}{H_\theta(x)} = (N-1) \frac{f_\theta(x)}{F_\theta(x)},$$

so the RHR order is inherited by the highest-rival distributions and gives the displayed highest-rival RHR order.

On every compact interval $I \subset V^\circ$, the formulas $H_\theta = F_\theta^{N-1}$ and $h_\theta = (N-1)F_\theta^{N-2}f_\theta$ inherit joint continuity. Compactness of $\Theta \times I$ gives $M_I < \infty$ and, because $H_\theta(x) > 0$ on the interior, $m_I > 0$.

Fix a compact rectangle $I \times C \subset V^\circ \times (0, \infty)$. Since ϕ is concave and $0 < \phi'(0) < \infty$, $0 < \phi'(y) \leq \phi'(0)$ for all $y \geq 0$. The density bound gives, for $(x, c) \in I \times C$,

$$|\phi'(H_\theta(x)c)h_\theta(x)c| \leq \phi'(0)M_I \sup C, \quad |\phi'(H_\theta(x)c)H_\theta(x)| \leq \phi'(0).$$

These bounds do not depend on θ and are μ -integrable. Joint continuity gives the required uniform domination, so differentiation under the μ -integral yields (7) and the derivatives are continuous. Positivity of Ψ_c follows from $\phi' > 0$ and $H_\theta(x) > 0$ for $x \in V^\circ$. $\square$

Lemma 2 (Necessity of the ODE). *Let β be the common bidding function in a non-decreasing symmetric equilibrium. Then*

$$\lim_{v \downarrow \underline{v}} \beta(v) = \underline{v}.$$

Moreover, $\beta(v) < v$ for every $v > \underline{v}$, β is strictly increasing and continuous on V° , and β is locally Lipschitz on V° . Hence β is absolutely continuous on compact subintervals of V° and differentiable almost everywhere. At every point of differentiability, β satisfies the ODE (1). Consequently, β is C^1 on V° and satisfies the ODE pointwise.

Proof. We prove the claim in five steps.

Step 1: Rule out bidding at one's valuation. Feasibility gives $\beta(v) \leq v$. Suppose $\beta(v) = v$ for some $v > \underline{v}$. Choose $t \in (\underline{v}, v)$. Since $\beta(t) \leq t < v$, a bidder with valuation v can deviate to $\beta(t)$. By monotonicity, this bid wins or ties whenever all $N-1$ opponents have valuations weakly below t , an event with probability $H_\theta(t) > 0$ under every $\theta \in \Theta$. Whenever the bidder wins or ties, the payoff is strictly positive. Hence $U(\beta(t), \beta, v) > \phi(0)$. At the bid $\beta(v) = v$, the bidder obtains zero payoff whether she wins, ties, or loses, so $U(\beta(v), \beta, v) = \phi(0)$. This contradicts optimality. Hence $\beta(v) < v$ for every $v > \underline{v}$.

Step 2: Rule out pooling. We show that a non-decreasing equilibrium must be strictly increasing. Suppose that β pools a nondegenerate interval of valuations at a bid $\hat{b}$. Let $I(\hat{b}) := \{x \in V : \beta(x) = \hat{b}\}$, let $\ell := \inf I(\hat{b})$, and let $h := \sup I(\hat{b})$, with $\ell < h$. Choose $v \in (\ell, h)$. By Step 1, $\hat{b} = \beta(v) < v$.

At $\hat{b}$, allocation is shared with every opponent whose valuation belongs to the pooling interval. Under model parameter θ , the allocation probability at the pooled bid is

$$P_{\theta}^{\text{pool}} = \sum_{k=0}^{N-1} \binom{N-1}{k} F_{\theta}(\ell)^{N-1-k} [F_{\theta}(h) - F_{\theta}(\ell)]^k \frac{1}{k+1}.$$

The binomial identity

$$P_{\theta}^{\text{pool}} = \frac{F_{\theta}(h)^N - F_{\theta}(\ell)^N}{N [F_{\theta}(h) - F_{\theta}(\ell)]}$$

shows that, because $F_{\theta}(h) > F_{\theta}(\ell)$,

$$P_{\theta}^{\text{pool}} < F_{\theta}(h)^{N-1} = H_{\theta}(h).$$

Thus

$$U(\hat{b}, \beta, v) = \int_{\Theta} \phi \left(P_{\theta}^{\text{pool}}(v - \hat{b}) \right) \mu(d\theta).$$

For $\varepsilon > 0$ small enough that $\hat{b} + \varepsilon < v$, a deviation to $\hat{b} + \varepsilon$ strictly beats every opponent in the pooling interval, and hence

$$U(\hat{b} + \varepsilon, \beta, v) \geq \int_{\Theta} \phi \left(H_{\theta}(h)(v - \hat{b} - \varepsilon) \right) \mu(d\theta).$$

At $\varepsilon = 0$, the latter expression is strictly larger than the payoff at $\hat{b}$. By continuity, the strict inequality persists for all sufficiently small $\varepsilon > 0$, contradicting optimality. Thus β is strictly increasing.

Lower-endpoint behavior. We next show that the equilibrium path attaches continuously to the lower boundary. Since β is non-decreasing and $0 \leq \beta(v) \leq v$, the right limit

$$\lim_{v \downarrow \underline{v}} \beta(v)$$

exists and is no larger than $\underline{v}$.

Suppose, to the contrary, that

$$\lim_{v \downarrow \underline{v}} \beta(v) < \underline{v}.$$

This strict inequality can arise only if $\underline{v} > 0$. There then exists $x \in V^\circ$ such that $\beta(x) < \underline{v}$. By the strict monotonicity established above, for every $v \in (\underline{v}, x)$,

$$U(\beta(v), \beta, v) = \int_{\Theta} \phi(H_\theta(v)[v - \beta(v)]) \mu(d\theta).$$

Joint continuity and compactness imply

$$\sup_{\theta \in \Theta} H_\theta(v) \longrightarrow 0 \quad \text{as } v \downarrow \underline{v}.$$

Since $v - \beta(v)$ is bounded,

$$U(\beta(v), \beta, v) \longrightarrow \phi(0).$$

The fixed deviation $\beta(x)$ is feasible for all $v > \underline{v}$, since $\beta(x) < \underline{v} < v$. Moreover,

$$U(\beta(x), \beta, v) = \int_{\Theta} \phi(H_\theta(x)[v - \beta(x)]) \mu(d\theta),$$

and therefore

$$U(\beta(x), \beta, v) \longrightarrow \int_{\Theta} \phi(H_\theta(x)[\underline{v} - \beta(x)]) \mu(d\theta) > \phi(0).$$

The strict inequality follows from $H_\theta(x) > 0$, $\underline{v} - \beta(x) > 0$, and the strict monotonicity of ϕ .

These two limits imply that every bidder with a valuation sufficiently close to $\underline{v}$ has a profitable deviation to $\beta(x)$, contradicting equilibrium. Hence

$$\lim_{v \downarrow \underline{v}} \beta(v) = \underline{v}.$$

Thus the equilibrium path has the continuous extension $\beta(\underline{v}) = \underline{v}$.

Step 3: Rule out jumps. Strict monotonicity alone is not enough; the bid function also cannot have gaps. Suppose that β has an upward jump at some $\hat{v} \in V^\circ$:

$$b^- := \lim_{x \uparrow \hat{v}} \beta(x) < \lim_{x \downarrow \hat{v}} \beta(x) =: b^+.$$

Choose $b^m \in (b^-, b^+)$. Let $v_n \downarrow \hat{v}$ and set $b_n := \beta(v_n)$. Then $b_n \rightarrow b^+$. Feasibility gives $b_n \leq v_n$, hence $b^+ \leq \hat{v}$. Therefore $b^m < v_n$ for all sufficiently large n .

Because β is strictly increasing, no valuation other than possibly $\hat{v}$ is mapped into (b^-, b^+) . Since every F_θ is continuous, the event that any opponent has valuation $\hat{v}$ has probability zero. Thus a deviation to b^m wins exactly when the highest rival valuation

is below $\hat{v}$:

$$U(b^m, \beta, v_n) = \int_{\Theta} \phi(H_{\theta}(\hat{v})(v_n - b^m)) \mu(d\theta).$$

At the equilibrium bid $b_n = \beta(v_n)$, strict monotonicity implies that a bidder with valuation v_n wins exactly when the highest rival valuation is below v_n :

$$U(b_n, \beta, v_n) = \int_{\Theta} \phi(H_{\theta}(v_n)(v_n - b_n)) \mu(d\theta).$$

Letting $n \rightarrow \infty$, these two payoffs converge to

$$\int_{\Theta} \phi(H_{\theta}(\hat{v})(\hat{v} - b^m)) \mu(d\theta) \quad \text{and} \quad \int_{\Theta} \phi(H_{\theta}(\hat{v})(\hat{v} - b^+)) \mu(d\theta),$$

respectively. Since $b^m < b^+$ and ϕ is strictly increasing, the first limit is strictly larger than the second. Thus, for all large n , a bidder with valuation v_n has a profitable deviation to b^m , a contradiction. Therefore β is continuous on V° . The same no-gap argument gives continuity at the finite upper endpoint $\bar{v}$.

Step 4: Establish local Lipschitz regularity. The purpose of this step is to convert incentive compatibility into a local slope bound. Fix $v_0 \in V^{\circ}$. Since β is continuous on V° and $\beta(v) < v$ for every $v > \underline{v}$, choose $r > 0$ such that

$$I := [v_0 - r, v_0 + r] \subset V^{\circ}, \quad m := \inf_{x \in I} \{x - \beta(x)\} > 0, \quad 2r < m.$$

Let $M := \sup_{x \in I} \{x - \beta(x)\}$ and use the compact rectangle $R := I \times [m/2, M + 2r]$. If $v, w \in I$ and $\Delta := w - v$, then $\Delta \leq 2r$, and hence $w - \beta(w) \in [m, M]$ and $v - \beta(v) + \Delta \in [m, M + 2r]$. The lower bound $m/2$ is a buffer for the nearby-valuation deviations used in the next step.

By [Lemma 1](#), Ψ_x and Ψ_c are continuous on R , and $\Psi_c > 0$ on R . Hence $\Lambda := \sup_R |\Psi_x| < \infty$ and $\alpha_0 := \inf_R \Psi_c > 0$. Take $v, w \in I$ with $v < w$, and write $\Delta := w - v$. Incentive compatibility for a bidder with valuation w , who can deviate to $\beta(v)$, gives

$$\Psi(w, w - \beta(w)) \geq \Psi(v, w - \beta(v)) = \Psi(v, v - \beta(v) + \Delta).$$

We compare both sides at the same second argument $w - \beta(w)$. Since

$$\Psi(w, w - \beta(w)) - \Psi(v, w - \beta(w)) = \int_v^w \Psi_x(y, w - \beta(w)) dy \leq \Lambda \Delta,$$

incentive compatibility implies

$$\Psi(v, v - \beta(v) + \Delta) - \Psi(v, w - \beta(w)) \leq \Lambda \Delta.$$

By the mean value theorem in the second argument, for some $\xi \in [m/2, M + 2r]$,

$$\Psi(v, v - \beta(v) + \Delta) - \Psi(v, w - \beta(w)) = \Psi_c(v, \xi) [\beta(w) - \beta(v)].$$

Therefore

$$0 \leq \beta(w) - \beta(v) \leq \frac{\Lambda}{\alpha_0} (w - v).$$

Thus β is Lipschitz on I . Since v_0 was arbitrary, β is locally Lipschitz on V° .

Step 5: Derive the ODE at differentiability points. Since β is locally Lipschitz, it is differentiable almost everywhere on V° . Fix a point $v \in V^\circ$ at which β is differentiable, and set $c := v - \beta(v) > 0$. For $h > 0$ sufficiently small, the bid $\beta(v + h)$ is feasible for a bidder with valuation v , because $\beta(v + h) \rightarrow \beta(v) < v$. Incentive compatibility at valuation v gives

$$\Psi(v, c) \geq \Psi(v + h, v - \beta(v + h)).$$

Incentive compatibility at valuation $v + h$, where the bidder can deviate to $\beta(v)$, gives

$$\Psi(v + h, v + h - \beta(v + h)) \geq \Psi(v, v + h - \beta(v)).$$

Taking one-sided difference quotients and letting $h \downarrow 0$ yields

$$0 \geq \Psi_x(v, c) - \Psi_c(v, c)\beta'(v) \quad \text{and} \quad 0 \leq \Psi_x(v, c) - \Psi_c(v, c)\beta'(v).$$

Hence

$$\beta'(v) = \frac{\Psi_x(v, c)}{\Psi_c(v, c)}.$$

Substituting $c = v - \beta(v)$ and (7) gives the ODE at every differentiability point.

Since β is absolutely continuous on compact subintervals and satisfies the ODE almost everywhere, it is an integral solution. The right-hand side of the ODE is continuous on the interior region $\{(v, b) : v \in V^\circ, b < v\}$. Therefore β is C^1 on V° and satisfies the ODE pointwise. $\square$

Lemma 3 (Existence). *Suppose [Assumption 1](#) holds. Then there exists a continuous function $\beta : V \rightarrow \mathbb{R}_+$ which is continuously differentiable on V° and solves the ODE (1). Moreover, it is strictly increasing on V° and exhibits bid shading, i.e. $\beta(v) < v$ for every $v > \underline{v}$.*

Proof. The proof has three steps. Step 1 constructs a local solution close to the initial value. Step 2 extends that solution to the whole support V . Step 3 verifies strict monotonicity.

Step 1. Local construction near the initial value. Set $v = \underline{v} + t$ for $t > 0$. Let

$$A_0 := \int_{\Theta} f_{\theta}(\underline{v})^{N-1} \mu(d\theta).$$

By the maintained density regularity, $f_{\theta}(\underline{v}) > 0$ for every $\theta \in \Theta$; hence $A_0 > 0$. By compactness of Θ and joint continuity of $(\theta, v) \mapsto f_{\theta}(v)$ on $\Theta \times V$,

$$F_{\theta}(\underline{v} + t) = \int_0^t f_{\theta}(\underline{v} + s) ds = f_{\theta}(\underline{v})t + o(t), \quad (8)$$

uniformly in θ as $t \downarrow 0$. Define, only for use in this proof,

$$\bar{H}(v) := \int_{\Theta} H_{\theta}(v) \mu(d\theta), \quad \bar{h}(v) := \int_{\Theta} h_{\theta}(v) \mu(d\theta).$$

Since $H_{\theta} = F_{\theta}^{N-1}$ and $h_{\theta} = (N-1)F_{\theta}^{N-2}f_{\theta}$ on V° ,

$$\bar{h}(v) = (N-1) \int_{\Theta} F_{\theta}(v)^{N-2} f_{\theta}(v) \mu(d\theta). \quad (9)$$

Consequently, the uniform expansion in [Eq. \(8\)](#) gives

$$\frac{\bar{H}(\underline{v} + t)}{t^{N-1}} = \int_{\Theta} \left(\frac{F_{\theta}(\underline{v} + t)}{t} \right)^{N-1} \mu(d\theta) \longrightarrow A_0, \quad (10)$$

and [Eq. \(9\)](#) gives

$$\frac{\bar{h}(\underline{v} + t)}{t^{N-2}} = (N-1) \int_{\Theta} \left(\frac{F_{\theta}(\underline{v} + t)}{t} \right)^{N-2} f_{\theta}(\underline{v} + t) \mu(d\theta) \longrightarrow (N-1)A_0. \quad (11)$$

In particular,

$$t \frac{\bar{h}(\underline{v} + t)}{\bar{H}(\underline{v} + t)} \longrightarrow N-1.$$

Using the approximation $\phi'(H_{\theta}(x)c) \approx \phi'(0)$ close to the initial value, we have

$$\beta'(v) = \frac{\Psi_x(v, v - \beta(v))}{\Psi_c(v, v - \beta(v))} \approx \frac{\bar{h}(v)(v - \beta(v))}{\bar{H}(v)} \approx (N-1) \frac{\underline{v} + t - \beta(\underline{v} + t)}{t}. \quad (12)$$

If locally $\beta(\underline{v} + t) \approx \underline{v} + at$, then [\(12\)](#) requires $a = (N-1)(1-a)$, and hence $a =$

$(N-1)/N$. Put $q := 1/N$ and therefore consider the ansatz

$$\beta(\underline{v} + t) = \underline{v} + \frac{N-1}{N}t - tz(t), \quad t > 0.$$

If z is bounded, this representation gives a continuous extension to the initial value. For this ansatz,

$$\underline{v} + t - \beta(\underline{v} + t) = t(q + z(t)), \quad \beta'(\underline{v} + t) = \frac{N-1}{N} - z(t) - tz'(t).$$

Substituting these expressions into the ODE gives

$$tz'(t) = -Nz(t) + R(t, z(t)), \quad (13)$$

where

$$R(t, z) := (N-1)(q + z)[1 - K(t, z)]$$

and

$$K(t, z) := \frac{\int_{\Theta} \phi'(H_{\theta}(\underline{v} + t)t(q + z)) \frac{h_{\theta}(\underline{v} + t)}{t^{N-2}} \mu(d\theta)}{(N-1) \int_{\Theta} \phi'(H_{\theta}(\underline{v} + t)t(q + z)) \frac{H_{\theta}(\underline{v} + t)}{t^{N-1}} \mu(d\theta)}.$$

From (13),

$$(t^N z(t))' = t^{N-1} R(t, z(t)).$$

Hence any solution for which the transformed function z is bounded must satisfy

$$z(t) = \frac{1}{t^N} \int_0^t \tau^{N-1} R(\tau, z(\tau)) d\tau, \quad t > 0. \quad (14)$$

Conversely, any continuous function satisfying (14) is C^1 on $(0, \delta]$ and solves (1) there. Therefore the local existence problem is reduced to finding a fixed point of the integral operator in (14).

We now prove existence by showing that the integral operator is a contraction. Fix $\eta \in (0, q)$. We work on the strip $z \in [-\eta, \eta]$, which ensures strictly positive surplus:

$$t(q + z) \geq t(q - \eta) > 0.$$

For $z \in [-\eta, \eta]$ and $\theta \in \Theta$,

$$0 \leq H_{\theta}(\underline{v} + t)t(q + z) \leq t(q + \eta).$$

Therefore, by continuity of ϕ' at zero,

$$\sup_{\theta \in \Theta, z \in [-\eta, \eta]} |\phi'(H_\theta(\underline{v} + t)t(q + z)) - \phi'(0)| \rightarrow 0 \quad \text{as } t \downarrow 0. \quad (15)$$

Let ε_t denote the supremum in (15). The errors from replacing ϕ' by $\phi'(0)$ in the numerator and the full denominator of $K(t, z)$ are bounded uniformly in z , respectively, by

$$\varepsilon_t \frac{\bar{h}(\underline{v} + t)}{t^{N-2}} \quad \text{and} \quad (N-1)\varepsilon_t \frac{\bar{H}(\underline{v} + t)}{t^{N-1}}.$$

By Eqs. (10) and (11), both bounds vanish. Hence the numerator and denominator of $K(t, z)$ converge uniformly in $z \in [-\eta, \eta]$ to the same strictly positive limit:

$$\phi'(0)(N-1)A_0 > 0.$$

Hence

$$\sup_{z \in [-\eta, \eta]} |K(t, z) - 1| \rightarrow 0 \quad \text{as } t \downarrow 0. \quad (16)$$

In particular, the denominator of K is bounded away from zero, uniformly in $z \in [-\eta, \eta]$, for all sufficiently small t .

We next obtain a Lipschitz constant in z . Since ϕ' is locally Lipschitz near zero, there is a constant $C > 0$ such that, for small t and all $z, z' \in [-\eta, \eta]$,

$$|\phi'(H_\theta(\underline{v} + t)t(q + z)) - \phi'(H_\theta(\underline{v} + t)t(q + z'))| \leq CH_\theta(\underline{v} + t)t|z - z'|.$$

For the scaled numerator of K , the resulting difference is bounded by

$$Ct|z - z'| \int_{\Theta} H_\theta(\underline{v} + t) \frac{h_\theta(\underline{v} + t)}{t^{N-2}} \mu(d\theta) \leq Ct|z - z'| \frac{\bar{h}(\underline{v} + t)}{t^{N-2}}.$$

For its scaled denominator, the difference is bounded by

$$C(N-1)t|z - z'| \int_{\Theta} \frac{H_\theta(\underline{v} + t)^2}{t^{N-1}} \mu(d\theta) \leq C(N-1)t|z - z'| \frac{\bar{H}(\underline{v} + t)}{t^{N-1}}.$$

The two ratios on the right are bounded by Eqs. (10) and (11), while the denominator of K is bounded away from zero and its numerator is uniformly bounded. Therefore the Lipschitz constant of $z \mapsto K(t, z)$ converges to zero as $t \downarrow 0$.

Set $R(0, z) = 0$, consistently with the uniform convergence above. Combining this

fact with (16), choose $\delta > 0$ small enough that, for all $t \in [0, \delta]$ and all $z, z' \in [-\eta, \eta]$,

$$|R(t, 0)| \leq \frac{N\eta}{4}, \quad |R(t, z) - R(t, z')| \leq \frac{N}{4}|z - z'|. \quad (17)$$

Define

$$X_\delta := \{z \in C([0, \delta]) : z(0) = 0, \|z\|_\infty \leq \eta\}$$

with the sup norm. This is a complete metric space. Define

$$(Tz)(t) = \begin{cases} \frac{1}{t^N} \int_0^t \tau^{N-1} R(\tau, z(\tau)) d\tau, & t > 0, \\ 0, & t = 0. \end{cases}$$

We show that T is a contraction from X_δ into itself. First, for $z \in X_\delta$, (17) implies

$$|R(t, z(t))| \leq |R(t, 0)| + \frac{N}{4}|z(t)| \leq \frac{N\eta}{2}.$$

Thus, for $t > 0$,

$$|(Tz)(t)| \leq \frac{1}{t^N} \int_0^t \tau^{N-1} \frac{N\eta}{2} d\tau = \frac{\eta}{2} \leq \eta.$$

Moreover, since $R(t, z) \rightarrow 0$ uniformly on $[-\eta, \eta]$,

$$|(Tz)(t)| \leq \frac{1}{N} \sup_{0 \leq \tau \leq t} |R(\tau, z(\tau))| \rightarrow 0,$$

so Tz is continuous at zero and $(Tz)(0) = 0$. Hence $T(X_\delta) \subseteq X_\delta$.

Second, for $z_1, z_2 \in X_\delta$, (17) gives

$$\begin{aligned} |(Tz_1)(t) - (Tz_2)(t)| &\leq \frac{1}{t^N} \int_0^t \tau^{N-1} \frac{N}{4} \|z_1 - z_2\|_\infty d\tau \\ &= \frac{1}{4} \|z_1 - z_2\|_\infty. \end{aligned}$$

Therefore T is a contraction. By Banach's fixed point theorem, there exists $z \in X_\delta$ satisfying (14). The fixed point is C^1 on $(0, \delta]$ and solves (13) there.

Define

$$\beta(\underline{v}) = \underline{v}, \quad \beta(\underline{v} + t) = \underline{v} + \frac{N-1}{N}t - tz(t), \quad t \in (0, \delta].$$

Since z is bounded and $z(0) = 0$, $\beta(\underline{v} + t) \rightarrow \underline{v}$ and

$$\underline{v} + t - \beta(\underline{v} + t) = t(q + z(t)) \geq t(q - \eta) > 0.$$

Thus the constructed branch is admissible, attaches continuously to the lower boundary, and solves the original bidding equation on $(\underline{v}, \underline{v} + \delta]$.

Step 2: Extend the local solution to the whole interior. Away from the boundary $\beta(v) = v$, compactness and joint continuity give the uniform interior bounds in [Lemma 1](#), and the right-hand side of the ODE is locally Lipschitz. Let $(\underline{v}, \hat{v})$ be the maximal interval on which the local solution has been extended and satisfies $\beta(v) < v$. Suppose $\hat{v} < \bar{v}$.

Since the ODE gives $\beta'(v) > 0$ whenever $\beta(v) < v$, and since $\beta(v) < v < \hat{v}$ for $v < \hat{v}$, the limit $\beta^* := \lim_{v \uparrow \hat{v}} \beta(v)$ exists and is finite. If $\beta^* < \hat{v}$, then $(\hat{v}, \beta^*)$ lies strictly inside the region $\{(v, b) : v > \underline{v}, b < v\}$. Local Lipschitzness of the ODE there allows continuation beyond $\hat{v}$, contradicting maximality. Thus the only possible obstruction is $\beta^* = \hat{v}$.

We rule this out. On a compact neighborhood of the interior point $\hat{v}$, the denominator of the ODE is bounded away from zero, while its numerator is bounded by a constant times $v - b$. Thus the ODE right-hand side extends continuously to the local diagonal with value zero. Hence, for $v < \hat{v}$ sufficiently close to $\hat{v}$ and $\beta(v)$ sufficiently close to v , $\beta'(v) < 1/2$. Then

$$\frac{d}{dv}(v - \beta(v)) > \frac{1}{2}$$

on a left neighborhood of $\hat{v}$. Integrating from v to $\hat{v}$ gives

$$0 - (v - \beta(v)) = \int_v^{\hat{v}} \frac{d}{dy}(y - \beta(y)) dy > 0,$$

which is impossible because $v - \beta(v) > 0$. Thus the solution cannot break down at any interior valuation, so $\hat{v} = \bar{v}$ and the solution extends uniquely to all of V° .

Define $\beta(\bar{v}) := \lim_{v \uparrow \bar{v}} \beta(v)$ by the monotone limit from below. This gives the continuous representative on all of V . Moreover, $\beta(\bar{v}) < \bar{v}$. Indeed, if equality held, the same uniform diagonal estimate on a compact left neighborhood of $\bar{v}$ would give $\beta'(v) < 1/2$ there, and integrating $(v - \beta(v))' > 1/2$ up to $\bar{v}$ would yield the same contradiction as above.

Step 3: Verify strict monotonicity. Since $v - \beta(v) > 0$ on V° , the ODE gives $\beta'(v) > 0$ for every $v \in V^\circ$. Thus β is strictly increasing on V° . $\square$

Lemma 4 (Verification and uniqueness). *Let β be a continuous function on V , C^1 on V° , satisfying $\beta(\underline{v}) = \underline{v}$, $\beta(v) < v$ for every $v > \underline{v}$, and the ODE on V° . Under [Assumptions 1 and 2](#), the profile in which every bidder uses β is a symmetric equilibrium.*

Moreover, there is at most one such non-decreasing solution.

Proof. We first prove the monotonicity property that makes the first-order condition sufficient. Fix $x \in V^\circ$. We show that

$$c \mapsto \frac{\Psi_x(x, c)}{\Psi_c(x, c)}$$

is strictly increasing on $(0, \infty)$. Using (7), a direct differentiation gives

$$\begin{aligned} \frac{d}{dc} \left[\frac{\Psi_x(x, c)}{\Psi_c(x, c)} \right] &= \frac{1}{\Psi_c(x, c)^2} \iint_{\Theta \times \Theta} \phi'(H_\theta(x)c) \phi'(H_\eta(x)c) h_\theta(x) H_\eta(x) \\ &\quad \times [1 + \rho(H_\eta(x)c) - \rho(H_\theta(x)c)] \mu(d\theta) \mu(d\eta). \end{aligned} \quad (18)$$

If alternative (ii) of **Assumption 2** holds, the bracketed term is bounded below by $1 - (\sup \rho - \inf \rho) > 0$, so the derivative in (18) is strictly positive.

Now suppose alternative (i) of **Assumption 2** holds. The part of (18) corresponding to the constant term 1 is strictly positive. It remains to show that the part involving $\rho(H_\eta(x)c) - \rho(H_\theta(x)c)$ is nonnegative. Symmetrizing that part gives

$$\begin{aligned} \frac{1}{2} \iint_{\Theta \times \Theta} \phi'(H_\theta(x)c) \phi'(H_\eta(x)c) H_\theta(x) H_\eta(x) \\ \times \left[\frac{h_\theta(x)}{H_\theta(x)} - \frac{h_\eta(x)}{H_\eta(x)} \right] [\rho(H_\eta(x)c) - \rho(H_\theta(x)c)] \mu(d\theta) \mu(d\eta). \end{aligned}$$

By **Assumption 1**, if $\theta \geq \eta$, then

$$\frac{h_\theta(x)}{H_\theta(x)} \geq \frac{h_\eta(x)}{H_\eta(x)} \quad \text{and} \quad H_\theta(x) \leq H_\eta(x).$$

Thus the two brackets are nonnegative when $\theta \geq \eta$ and nonpositive when $\eta \geq \theta$. Hence their product is always nonnegative. Therefore the derivative in (18) is strictly positive.

We now verify that β is a best response. Since β solves the ODE and $\beta(v) < v$, its derivative is strictly positive, so β is strictly increasing on V° . Fix a bidder with valuation $v > \underline{v}$. Any bid in the image of β can be written as $b = \beta(x)$ for some $x \in V^\circ$. Such a bid wins exactly when the highest rival valuation is below x , so

$$U(\beta(x), \beta, v) = \Psi(x, v - \beta(x)).$$

Differentiating with respect to x ,

$$\frac{d}{dx} \Psi(x, v - \beta(x)) = \Psi_x(x, v - \beta(x)) - \Psi_c(x, v - \beta(x)) \beta'(x).$$

Because β solves the ODE,

$$\beta'(x) = \frac{\Psi_x(x, x - \beta(x))}{\Psi_c(x, x - \beta(x))}.$$

Thus the sign of the derivative is the sign of

$$\frac{\Psi_x(x, v - \beta(x))}{\Psi_c(x, v - \beta(x))} - \frac{\Psi_x(x, x - \beta(x))}{\Psi_c(x, x - \beta(x))}.$$

By the monotonicity just proved, this sign is the sign of $(v - \beta(x)) - (x - \beta(x)) = v - x$. Therefore $U(\beta(x), \beta, v)$ is increasing in x for $x < v$ and decreasing in x for $x > v$. Since β is strictly increasing, $\beta(v)$ is the unique maximizer among bids in the image of β .

Bids below $\beta(\underline{v}) = \underline{v}$ win only on null opponent profiles and yield $\phi(0)$, while $U(\beta(v), \beta, v) = \Psi(v, v - \beta(v)) > \phi(0)$ for every $v > \underline{v}$. Such lower bids cannot be optimal. The image of β has the finite upper endpoint $\beta(\bar{v})$. Bids above that endpoint win with probability one and yield $\phi(v - b)$, which is decreasing in b . These bids cannot improve upon the decreasing limit of payoffs along the image above $\beta(v)$. Hence $\beta(v)$ is a best response for every $v > \underline{v}$. At $v = \underline{v}$, the boundary value is the continuous representative. Thus the profile in which every bidder uses β is a symmetric equilibrium.

This establishes verification: conditional on rivals using the candidate strategy β , for every interior valuation v , the unique optimal bid is $\beta(v)$. This does not yet exclude a different candidate strategy that is also a fixed point of best responses. It remains to show that the boundary-value problem admits at most one admissible non-decreasing solution.

Let β_1 and β_2 be two non-decreasing C^1 solutions satisfying the stated boundary condition. Define $d(v) := (v - \beta_1(v)) - (v - \beta_2(v))$. By the ODE,

$$d'(v) = - \left[\frac{\Psi_x(v, v - \beta_1(v))}{\Psi_c(v, v - \beta_1(v))} - \frac{\Psi_x(v, v - \beta_2(v))}{\Psi_c(v, v - \beta_2(v))} \right].$$

The bracket has the same sign as $d(v)$ by the monotonicity proved above. Hence $d(v)d'(v) \leq 0$, so $(d(v)^2)' \leq 0$ on V° . Since both solutions have the same continuous boundary representative, $\lim_{v \downarrow \underline{v}} d(v) = 0$. Therefore $d(v)^2 = 0$ for every $v \in V^\circ$, and the two solutions coincide. Thus the admissible non-decreasing solution of the boundary-value problem is unique. $\square$

Proof of Theorem 1. Existence follows from Lemma 3. By Lemma 4, the profile induced by the solution constructed in Lemma 3 is a symmetric equilibrium.

Conversely, let β be the common bidding function in any non-decreasing symmetric

equilibrium. By [Lemma 2](#), β is locally Lipschitz on V° and satisfies the ODE almost everywhere. Since the right-hand side of the ODE is continuous on the interior region where $v > \underline{v}$ and $\beta(v) < v$, β is an integral solution and hence is C^1 on V° and satisfies the ODE pointwise.

[Lemma 2](#) shows that every non-decreasing symmetric equilibrium has the same continuous lower-endpoint extension, $\beta(\underline{v}) = \underline{v}$. It therefore satisfies the same boundary-value problem as the solution constructed in [Lemma 3](#). [Lemma 4](#) implies that the two bidding paths coincide. Hence the non-decreasing symmetric equilibrium is unique. $\square$

Proof of [Corollary 1](#). Under ambiguity neutrality, $\phi'(x) = 1$, and the highest-rival CDF satisfies

$$\int_{\Theta} H_{\theta}(v) \mu(d\theta) = \int_{\Theta} F_{\theta}(v)^{N-1} \mu(d\theta) = F_{N-1}(v)^{N-1}.$$

Substituting $\phi'(x) = 1$ into the equilibrium ODE therefore yields the standard Bayesian FPA ODE under the individual-value distribution F_{N-1} . $\square$

Appendix B Bidding Comparative Statics and Benchmarks

Proof of [Theorem 2](#). Since A is more ambiguity averse than B , there exists an increasing concave function χ such that

$$\phi_A = \chi \circ \phi_B.$$

Because ϕ_A and ϕ_B are continuously differentiable and $\phi'_B > 0$, χ is differentiable on the relevant range and

$$\phi'_A(x) = \chi'(\phi_B(x)) \phi'_B(x).$$

For $k \in \{A, B\}$, define

$$Q_k(v, c) := c \frac{\int_{\Theta} \phi'_k(H_{\theta}(v)c) h_{\theta}(v) \mu(d\theta)}{\int_{\Theta} \phi'_k(H_{\theta}(v)c) H_{\theta}(v) \mu(d\theta)},$$

and define the probability measure

$$\nu_{k,v,c}(d\theta) := \frac{\phi'_k(H_{\theta}(v)c) H_{\theta}(v)}{\int_{\Theta} \phi'_k(H_{\eta}(v)c) H_{\eta}(v) \mu(d\eta)} \mu(d\theta).$$

The definition of $\nu_{k,v,c}$ allows us to write

$$Q_k(v, c) = c \int_{\Theta} \frac{h_{\theta}(v)}{H_{\theta}(v)} \nu_{k,v,c}(d\theta).$$

Thus, $Q_k(v, c)/c$ is the ambiguity-adjusted expectation of the candidate highest-rival reverse hazard rate.

We next compare the two ambiguity-adjusted measures. Their likelihood ratio is

$$\begin{aligned} \frac{d\nu_{A,v,c}}{d\nu_{B,v,c}}(\theta) &= \frac{\int_{\Theta} \phi'_B(H_\eta(v)c) H_\eta(v) \mu(d\eta) \phi'_A(H_\theta(v)c)}{\int_{\Theta} \phi'_A(H_\eta(v)c) H_\eta(v) \mu(d\eta) \phi'_B(H_\theta(v)c)} \\ &= C(v, c) \chi'(\phi_B(H_\theta(v)c)), \end{aligned}$$

where $C(v, c) > 0$ does not depend on θ .

Fix $v \in V^\circ$. If $\theta' \geq \theta$, the RHR order gives

$$\frac{h_{\theta'}(v)}{H_{\theta'}(v)} = (N-1) \frac{f_{\theta'}(v)}{F_{\theta'}(v)} \geq (N-1) \frac{f_\theta(v)}{F_\theta(v)} = \frac{h_\theta(v)}{H_\theta(v)}.$$

The same RHR order, together with the common upper endpoint, implies

$$F_{\theta'}(v) \leq F_\theta(v),$$

and hence

$$H_{\theta'}(v) \leq H_\theta(v).$$

Indeed,

$$\log F_{\theta'}(v) - \log F_\theta(v) = - \int_v^{\bar{v}} \left[\frac{f_{\theta'}(t)}{F_{\theta'}(t)} - \frac{f_\theta(t)}{F_\theta(t)} \right] dt \leq 0.$$

Since χ is concave, χ' is non-increasing, while ϕ_B is increasing. It follows from $H_{\theta'}(v) \leq H_\theta(v)$ that

$$\chi'(\phi_B(H_{\theta'}(v)c)) \geq \chi'(\phi_B(H_\theta(v)c)).$$

Therefore,

$$\theta \mapsto \frac{d\nu_{A,v,c}}{d\nu_{B,v,c}}(\theta)$$

is weakly increasing. In other words, $\nu_{A,v,c}$ dominates $\nu_{B,v,c}$ in the monotone likelihood-ratio order and hence also in the first-order stochastic dominance order.

At the same time,

$$\theta \mapsto \frac{h_\theta(v)}{H_\theta(v)}$$

is weakly increasing. First-order stochastic dominance therefore gives

$$\int_{\Theta} \frac{h_\theta(v)}{H_\theta(v)} \nu_{A,v,c}(d\theta) \geq \int_{\Theta} \frac{h_\theta(v)}{H_\theta(v)} \nu_{B,v,c}(d\theta).$$

Multiplying by $c > 0$, we obtain

$$Q_A(v, c) \geq Q_B(v, c) \quad \text{for every } v \in V^\circ \text{ and } c > 0.$$

Let

$$s_k(v) := v - \beta_k(v), \quad k \in \{A, B\}.$$

The equilibrium ODEs can be written as

$$s'_k(v) = 1 - Q_k(v, s_k(v)).$$

The monotonicity property established in the verification argument implies that

$$c \mapsto Q_B(v, c)$$

is strictly increasing.

Suppose that $s_A(v) > s_B(v)$ at some valuation. At every such valuation,

$$\begin{aligned} (s_A - s_B)'(v) &= -Q_A(v, s_A(v)) + Q_B(v, s_B(v)) \\ &\leq -Q_B(v, s_A(v)) + Q_B(v, s_B(v)) \\ &< 0. \end{aligned}$$

The first inequality follows from $Q_A(v, c) \geq Q_B(v, c)$ at the common surplus $c = s_A(v)$; the second follows from $s_A(v) > s_B(v)$ and the strict increase of $Q_B(v, \cdot)$.

Now consider a connected component of

$$\{v \in V^\circ : s_A(v) > s_B(v)\}.$$

At its left boundary, continuity and the common lower-endpoint condition imply $s_A - s_B = 0$, whereas inside the component $s_A - s_B$ is strictly decreasing. It therefore cannot become strictly positive. Hence no such component exists, and

$$s_A(v) \leq s_B(v) \quad \text{for every } v \in V.$$

Equivalently,

$$\beta_A(v) \geq \beta_B(v) \quad \text{for every } v \in V.$$

□

Proof of Proposition 1. Let $s(v) := v - \beta(v)$ and write the equilibrium coefficient as

$$r(v, s) := \frac{\int_{\Theta} \phi'(H_{\theta}(v)s) h_{\theta}(v) \mu(d\theta)}{\int_{\Theta} \phi'(H_{\theta}(v)s) H_{\theta}(v) \mu(d\theta)}.$$

It is a weighted average of $h_{\theta}(v)/H_{\theta}(v)$. Thus, with $H^{\pm} = (F^{\pm})^{N-1}$,

$$\frac{h^{-}(v)}{H^{-}(v)} \leq r(v, s) \leq \frac{h^{+}(v)}{H^{+}(v)}.$$

Since $s' = 1 - r(v, s)s$, these inequalities imply

$$(H^{+}(v)s(v))' \geq H^{+}(v), \quad (H^{-}(v)s(v))' \leq H^{-}(v).$$

Integrating from $\underline{v}$ and using the common boundary value gives

$$\frac{\int_{\underline{v}}^v H^{+}(t) dt}{H^{+}(v)} \leq s(v) \leq \frac{\int_{\underline{v}}^v H^{-}(t) dt}{H^{-}(v)}.$$

Subtracting these surplus inequalities from v proves the stated bid bounds. $\square$

Proof of Theorem 3. By the Arrow–Pratt characterization of comparative smooth ambiguity aversion and Theorem 2, $\beta_{k+1}(v) \geq \beta_k(v)$ for every $v \in V$, while Proposition 1 gives $\beta_k(v) \leq \beta^{\min}(v)$. Hence, for every $v \in V$, the pointwise limit $\beta^*(v) := \lim_{k \rightarrow \infty} \beta_k(v)$ exists and satisfies $\beta^*(v) \leq \beta^{\min}(v)$.

We next derive the limiting incentive constraints. Fix $v, x \in V$ such that $\beta^{\min}(x) \leq v$. Then $\beta_k(x) \leq v$, so a bidder with valuation v can mimic one with valuation x by bidding $\beta_k(x)$. Fix j . For every $k \geq j$, equilibrium optimality and monotonicity of ϕ_k imply

$$\begin{aligned} & \int_{\Theta} \phi_k(H_{\theta}(v)[v - \beta_j(v)]) \mu(d\theta) \\ & \geq \int_{\Theta} \phi_k(H_{\theta}(v)[v - \beta_k(v)]) \mu(d\theta) \\ & \geq \int_{\Theta} \phi_k(H_{\theta}(x)[v - \beta_k(x)]) \mu(d\theta) \\ & \geq \int_{\Theta} \phi_k(H_{\theta}(x)[v - \beta^*(x)]) \mu(d\theta). \end{aligned}$$

The first and last payoff profiles in this chain are fixed as k varies. **Klibanoff et al.**

(2005, Proposition 3) therefore imply

$$\operatorname{ess\,inf}_{\theta \sim \mu} H_\theta(v)[v - \beta_j(v)] \geq \operatorname{ess\,inf}_{\theta \sim \mu} H_\theta(x)[v - \beta^*(x)].$$

By compactness of Θ , full support of μ , and continuity in θ , the essential infima are minima. Since both surplus terms are nonnegative and $\min_{\theta \in \Theta} H_\theta(y) = H^+(y)$, we obtain

$$H^+(v)[v - \beta_j(v)] \geq H^+(x)[v - \beta^*(x)].$$

Letting $j \rightarrow \infty$ gives

$$H^+(v)[v - \beta^*(v)] \geq H^+(x)[v - \beta^*(x)] \quad \text{whenever } \beta^{\min}(x) \leq v. \quad (19)$$

Fix $v \in V^\circ$. If $w < v$ is sufficiently close to v , then $\beta^{\min}(v) < w < v$. Applying Eq. (19) first to a bidder with valuation v bidding as one with valuation w , and then vice versa, yields

$$\begin{aligned} H^+(w)(v - w) &\leq H^+(v)[v - \beta^*(v)] - H^+(w)[w - \beta^*(w)] \\ &\leq H^+(v)(v - w). \end{aligned} \quad (20)$$

Similarly, if $w > v$ is sufficiently close to v , then $\beta^{\min}(w) < v$, and the two incentive constraints give

$$\begin{aligned} H^+(v)(w - v) &\leq H^+(w)[w - \beta^*(w)] - H^+(v)[v - \beta^*(v)] \\ &\leq H^+(w)(w - v). \end{aligned} \quad (21)$$

By continuity of H^+ , Eqs. (20) and (21) imply

$$\frac{d}{dv} \{H^+(v)[v - \beta^*(v)]\} = H^+(v).$$

Moreover, $\lim_{v \downarrow \underline{v}} H^+(v)[v - \beta^*(v)] = 0$, because $H^+(v) \rightarrow 0$ and the surplus is bounded. Therefore

$$H^+(v)[v - \beta^*(v)] = \int_{\underline{v}}^v H^+(t) dt.$$

It follows that

$$\beta^*(v) = v - \frac{\int_{\underline{v}}^v H^+(t) dt}{H^+(v)} = \beta^{\min}(v)$$

for every $v \in V^\circ$. Equality at the lower endpoint follows from the normalization. Equality at the upper endpoint follows from monotonicity of β^* , the inequality $\beta^* \leq \beta^{\min}$, and continuity of $\beta^{\min}$. $\square$

Appendix C CRAA and Auction-Format Comparisons

C.1 Constant Relative Risk Aversion (CRRA) Analogy

Consider the symmetric equilibrium of the first-price auction with risk-averse bidders in the SEU framework.

Lemma 5 (CRRA benchmark). *Let bidders share the candidate belief F . Let $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ be continuous, strictly increasing, and concave, with $u(0) = 0$, and continuously differentiable with $u'(x) > 0$ for $x > 0$. Any differentiable increasing symmetric equilibrium of the standard Bayesian IPV first-price auction satisfies*

$$\beta'(v) = (N-1) \frac{f(v)u(v-\beta(v))}{F(v)u'(v-\beta(v))},$$

with $\beta(\underline{v}) = \underline{v}$.

If $u(x) = x^\gamma$ with $0 < \gamma \leq 1$, then

$$\beta'(v) = \frac{N-1}{\gamma} \frac{f(v)}{F(v)} (v - \beta(v)).$$

Consequently, under the standard conditions ensuring uniqueness of the increasing symmetric Bayesian equilibrium, the CRRA bidding function coincides with the risk-neutral equilibrium under the distribution $F_\gamma := F^{1/\gamma}$.

C.2 CRAA Coefficient and Equilibrium Proofs

For every compact interval $I \subset V^\circ$, compactness and joint continuity give

$$\min_{(\theta, v) \in \Theta \times I} H_\theta(v) > 0,$$

so the negative-power and logarithmic expressions used below are well defined at interior valuations.

For a bounded nonnegative measurable collection $X = \{X_\theta\}_{\theta \in \Theta}$ and $p \in \mathbb{R}$, define the generalized mean under μ by

$$M_p(X) := \begin{cases} \left(\int_{\Theta} X_\theta^p \mu(d\theta) \right)^{1/p}, & p \neq 0, \\ \exp \left(\int_{\Theta} \log X_\theta \mu(d\theta) \right), & p = 0. \end{cases}$$

For $p \leq 0$, the formula for a collection that may vanish is understood by its monotone

continuous extension from strictly positive collections, $M_p(X) := \lim_{\varepsilon \downarrow 0} M_p(X + \varepsilon)$. In particular, this convention includes the geometric-mean case at zero and the corresponding extended-value convention for negative powers.

The CRAA certainty equivalent is therefore

$$CE_a(\{X_\theta\}_{\theta \in \Theta}) = M_{1-a}(X) = \begin{cases} \left(\int_{\Theta} X_\theta^{1-a} \mu(d\theta) \right)^{1/(1-a)}, & a \neq 1, \\ \exp \left(\int_{\Theta} \log X_\theta \mu(d\theta) \right), & a = 1. \end{cases}$$

Applying this mean to the highest-rival CDFs under the candidate beliefs gives

$$H_a(v) := M_{1-a}(\{H_\theta(v)\}_{\theta \in \Theta}) = \begin{cases} \left(\int_{\Theta} H_\theta(v)^{1-a} \mu(d\theta) \right)^{1/(1-a)}, & a \neq 1, \\ \exp \left(\int_{\Theta} \log H_\theta(v) \mu(d\theta) \right), & a = 1. \end{cases}$$

Since $H_\theta = F_\theta^{N-1}$, the associated individual effective CDF satisfies

$$F_a(v) := H_a(v)^{1/(N-1)} = M_{(N-1)(1-a)}(\{F_\theta(v)\}_{\theta \in \Theta}).$$

Equivalently,

$$F_a(v) = \begin{cases} \left(\int_{\Theta} F_\theta(v)^{(N-1)(1-a)} \mu(d\theta) \right)^{1/[(N-1)(1-a)]}, & a \neq 1, \\ \exp \left(\int_{\Theta} \log F_\theta(v) \mu(d\theta) \right), & a = 1. \end{cases}$$

We next verify the required distributional properties. Generalized means are monotone and satisfy

$$\min_{\theta \in \Theta} H_\theta(v) \leq H_a(v) \leq \max_{\theta \in \Theta} H_\theta(v).$$

Hence H_a is non-decreasing. Joint continuity and compactness imply uniform convergence of $H_\theta(v)$ to zero as $v \downarrow \underline{v}$ and to one as $v \uparrow \bar{v}$. The displayed bounds therefore give $H_a(\underline{v}) = 0$, $H_a(\bar{v}) = 1$, and continuity at both endpoints. On V° , differentiation under the integral sign gives

$$H'_a(v) = H_a(v)^a \int_{\Theta} H_\theta(v)^{-a} h_\theta(v) \mu(d\theta) > 0,$$

where the same formula follows by differentiating the logarithmic expression when $a = 1$. Consequently, H_a is a continuous CDF on V and is continuously differentiable with

strictly positive density on V° . Since $F_a = H_a^{1/(N-1)}$, the same endpoint, monotonicity, and interior regularity properties hold for F_a .

Proof of Proposition 2.

$$CE_a(\{H_\theta(x)[v - \beta(x)]\}_{\theta \in \Theta}) = H_a(x)[v - \beta(x)] = F_a(x)^{N-1}[v - \beta(x)].$$

Write $c := v - \beta(x)$. For $a \neq 1$, homogeneity of the CRAA certainty equivalent gives

$$\begin{aligned} CE_a(\{H_\theta(x)c\}_{\theta \in \Theta}) &= \left(\int_{\Theta} [H_\theta(x)c]^{1-a} \mu(d\theta) \right)^{1/(1-a)} \\ &= c \left(\int_{\Theta} H_\theta(x)^{1-a} \mu(d\theta) \right)^{1/(1-a)} = cH_a(x). \end{aligned}$$

For $a = 1$, the same identity follows from

$$\exp \left(\int_{\Theta} \log[H_\theta(x)c] \mu(d\theta) \right) = c \exp \left(\int_{\Theta} \log H_\theta(x) \mu(d\theta) \right) = cH_a(x),$$

with the zero-surplus case obtained by continuity. Since $H_a = F_a^{N-1}$, this proves the displayed best-response identity. The identity holds for every deviation indexed by x , so the CRAA best-response problem is exactly the risk-neutral best-response problem with i.i.d. distribution F_a . Their symmetric equilibrium bidding functions therefore coincide. $\square$

C.3 Bidder-Preference Proofs

We use the generalized mean M_p defined in [Appendix C.2](#). For $p \leq 1$, M_p is homogeneous of degree one and concave on the positive cone and hence is superadditive:

$$M_p(X + Y) \geq M_p(X) + M_p(Y).$$

For completeness, we verify these properties below.

Proof of concavity and superadditivity. For completeness, consider first a finite support $\{1, \dots, n\}$ with probabilities $\pi_i > 0$. For $p \neq 0$ write

$$M_p(x) = \left(\sum_i \pi_i x_i^p \right)^{1/p}, \quad x_i > 0.$$

A direct differentiation gives, for any vector z ,

$$z' D^2 M_p(x) z = (1 - p) M_p(x)^{1-2p} \left[\left(\sum_i \pi_i x_i^{p-1} z_i \right)^2 - \left(\sum_i \pi_i x_i^p \right) \left(\sum_i \pi_i x_i^{p-2} z_i^2 \right) \right].$$

By Cauchy's inequality,

$$\left(\sum_i \pi_i x_i^{p-1} z_i \right)^2 \leq \left(\sum_i \pi_i x_i^p \right) \left(\sum_i \pi_i x_i^{p-2} z_i^2 \right).$$

Since $p \leq 1$, the factor $1 - p$ is nonnegative, so $z' D^2 M_p(x) z \leq 0$. Thus M_p is concave. The case $p = 0$ follows by taking the limit $p \rightarrow 0$, which yields the geometric mean. The general case follows by finite-support approximation.

Homogeneity is immediate from the definition: $M_p(cX) = c M_p(X)$ for $c > 0$. Concavity and homogeneity imply superadditivity:

$$\frac{1}{2} M_p(X + Y) = M_p\left(\frac{X + Y}{2}\right) \geq \frac{1}{2} M_p(X) + \frac{1}{2} M_p(Y).$$

Multiplying by 2 yields the displayed inequality. $\square$

Proof of Proposition 6. Let $p := 1 - a$ and let $s_a(v) := v - \beta_a(v)$ denote the equilibrium FPA surplus. By Proposition 2, the effective highest-rival CDF is $H_a(v) = M_p(H_\theta(v))$, where $H_\theta(v)$ is viewed as the random variable $\theta \mapsto H_\theta(v)$ on (Θ, μ) . The equilibrium characterization gives

$$s_a(v) = \frac{\int_v^v H_a(t) dt}{H_a(v)}.$$

Since the CRAA certainty equivalent is M_p and M_p is homogeneous,

$$\begin{aligned} CE_a^{\text{FPA}}(v) &= M_p(H_\theta(v) s_a(v)) \\ &= s_a(v) H_a(v) \\ &= \int_v^v H_a(t) dt \\ &= \int_v^v M_p(H_\theta(t)) dt. \end{aligned}$$

Under truthful bidding in the SPA, let

$$Y_{-i} := \max_{j \neq i} v_j$$

denote the highest rival valuation under candidate belief F_θ . The payoff under candidate belief F_θ is

$$A_\theta(v) = \mathbb{E}_\theta[(v - Y_{-i})_+] = \int_{\underline{v}}^v H_\theta(t) dt.$$

Hence

$$CE_a^{\text{SPA}}(v) = M_p \left(\int_{\underline{v}}^v H_\theta(t) dt \right).$$

Since $p \leq 1$, superadditivity of the generalized mean, applied first to finite sums and then to the integral by approximation, gives

$$CE_a^{\text{SPA}}(v) \geq \int_{\underline{v}}^v M_p(H_\theta(t)) dt = CE_a^{\text{FPA}}(v).$$

Because ϕ is strictly increasing, $U^{\text{SPA}}(v) \geq U^{\text{FPA}}(v)$. $\square$

Proof of Proposition 7. Let $s(v) := v - \beta(v)$ and $A_\theta(v) := \int_{\underline{v}}^v H_\theta(t) dt$, and define the utility difference

$$D(v) := \int_{\Theta} [\phi(H_\theta(v)s(v)) - \phi(A_\theta(v))] \mu(d\theta).$$

It suffices to prove $D(v) \geq 0$.

Since $A'_\theta(v) = H_\theta(v)$ and the equilibrium surplus equation is

$$s'(v) = 1 - s(v) \frac{\int_{\Theta} \phi'(H_\theta(v)s(v)) h_\theta(v) \mu(d\theta)}{\int_{\Theta} \phi'(H_\theta(v)s(v)) H_\theta(v) \mu(d\theta)},$$

differentiating D makes the terms involving h_θ cancel and yields

$$D'(v) = \int_{\Theta} H_\theta(v) [\phi'(H_\theta(v)s(v)) - \phi'(A_\theta(v))] \mu(d\theta). \quad (22)$$

For each θ , write

$$x_\theta := H_\theta(v)s(v), \quad y_\theta := A_\theta(v), \quad Z_\theta := \phi(x_\theta) - \phi(y_\theta),$$

and define

$$\kappa(x, y) := \begin{cases} -\frac{\phi'(x) - \phi'(y)}{\phi(x) - \phi(y)}, & x \neq y, \\ \alpha(x), & x = y. \end{cases}$$

Then

$$\phi'(x) - \phi'(y) = -\kappa(x, y)[\phi(x) - \phi(y)].$$

For $x \neq y$, the fundamental theorem of calculus and $-\phi'' = \alpha\phi'$ give

$$\kappa(x, y) = \frac{\int_{\min\{x, y\}}^{\max\{x, y\}} \alpha(t)\phi'(t) dt}{\int_{\min\{x, y\}}^{\max\{x, y\}} \phi'(t) dt}.$$

Under IAAA, this weighted average is weakly increasing in each argument.

We next use the RHR order. If $\theta \geq \eta$, then $H_\theta(v) \leq H_\eta(v)$ and, for $t \leq v$,

$$\frac{H_\theta(t)}{H_\theta(v)} = \exp\left(-\int_t^v \frac{h_\theta(z)}{H_\theta(z)} dz\right) \leq \frac{H_\eta(t)}{H_\eta(v)}.$$

Integrating gives

$$\frac{A_\theta(v)}{H_\theta(v)} \leq \frac{A_\eta(v)}{H_\eta(v)}. \quad (23)$$

Consequently, as $H_\theta(v)$ increases along the RHR order, both x_θ and y_θ increase. By IAAA, $\kappa(x_\theta, y_\theta)$ increases as well, and therefore $w_\theta := H_\theta(v)\kappa(x_\theta, y_\theta)$ is increasing in $H_\theta(v)$.

Moreover,

$$Z_\theta \geq 0 \iff s(v) \geq \frac{A_\theta(v)}{H_\theta(v)}.$$

By [Eq. \(23\)](#), Z_θ single-crosses from nonnegative to nonpositive as $H_\theta(v)$ increases. Combining this observation with [Eq. \(22\)](#) gives

$$D'(v) = -\int_{\Theta} w_\theta Z_\theta \mu(d\theta).$$

Suppose $D(v) = \int_{\Theta} Z_\theta \mu(d\theta) \leq 0$. Choose a value w_0 at the single-crossing threshold. Since w_θ is increasing while Z_θ single-crosses from nonnegative to nonpositive,

$$\int_{\Theta} w_\theta Z_\theta \mu(d\theta) = \int_{\Theta} (w_\theta - w_0) Z_\theta \mu(d\theta) + w_0 D(v) \leq 0.$$

Thus $D'(v) \geq 0$ whenever $D(v) \leq 0$. Finally, $D(\underline{v}) = 0$. If the open set $\{v : D(v) < 0\}$ had a connected component, D would be weakly increasing throughout that component from a zero left-boundary value, a contradiction. Hence $D(v) \geq 0$ for every $v \in V$, proving $U^{\text{FPA}}(v) \geq U^{\text{SPA}}(v)$. $\square$

Appendix D Seller-Revenue Results

Proof of Proposition 3. Recall from Corollary 1 that

$$F_{N-1}(v)^{N-1} = \int_{\Theta} F_{\theta}(v)^{N-1} \mu(d\theta),$$

and that β_0 is the equilibrium bid in the standard Bayesian auction with individual valuations distributed according to F_{N-1} . In that auction, standard revenue equivalence gives

$$\int_V \beta_0(v) d(F_{N-1}(v)^N) = \mathbb{E}_{F_{N-1}} [v_{(2)}].$$

Denote by $R_{\text{id}}^{\text{FPA}}$ the revenue of the first-price auction under ambiguity neutrality. We can thus write the difference between the revenue of the first-price and the second-price auction as follows.

$$\begin{aligned} R^{\text{SPA}} - R_{\text{id}}^{\text{FPA}} &= R^{\text{SPA}} - \int_V \beta_0(v) d\left(\int_{\Theta} F_{\theta}(v)^N \mu(d\theta)\right) \\ &= \left[R^{\text{SPA}} - \mathbb{E}_{F_{N-1}} [v_{(2)}] \right] \\ &\quad + \left[\int_V \beta_0(v) d(F_{N-1}(v)^N) - \int_V \beta_0(v) d\left(\int_{\Theta} F_{\theta}(v)^N \mu(d\theta)\right) \right]. \end{aligned}$$

We first compute

$$R^{\text{SPA}} - \mathbb{E}_{F_{N-1}} [v_{(2)}],$$

where the latter expectation is taken in the standard Bayesian auction in which all N valuations are independently distributed according to F_{N-1} .

For each fixed θ , let G_{θ} denote the conditional CDF of the second-highest valuation. By definition,

$$G_{\theta}(v) = \int_{V^N} \mathbf{1}\{v_{(2)} \leq v\} \prod_{i=1}^N dF_{\theta}(v_i).$$

The event $v_{(2)} \leq v$ occurs if either all valuations are at most v , or exactly one valuation exceeds v . Conditional independence therefore gives

$$\begin{aligned} G_{\theta}(v) &= F_{\theta}(v)^N + N[1 - F_{\theta}(v)]F_{\theta}(v)^{N-1} \\ &= NF_{\theta}(v)^{N-1} - (N-1)F_{\theta}(v)^N. \end{aligned}$$

Consequently,

$$\begin{aligned}\mathbb{E}^\theta [v_{(2)}] &= \int_V v \, dG_\theta(v) \\ &= \bar{v} - \int_{\underline{v}}^{\bar{v}} [NF_\theta(v)^{N-1} - (N-1)F_\theta(v)^N] \, dv.\end{aligned}$$

Averaging this identity over θ gives an expansion of [Eq. \(5\)](#):

$$\begin{aligned}R^{\text{SPA}} &= \int_{\Theta} \left\{ \bar{v} - \int_{\underline{v}}^{\bar{v}} [NF_\theta(v)^{N-1} - (N-1)F_\theta(v)^N] \, dv \right\} \mu(d\theta) \\ &= \bar{v} - \int_{\underline{v}}^{\bar{v}} \left[N \int_{\Theta} F_\theta(v)^{N-1} \mu(d\theta) - (N-1) \int_{\Theta} F_\theta(v)^N \mu(d\theta) \right] \, dv.\end{aligned}$$

In the standard Bayesian auction with valuation distribution F_{N-1} , the CDF of the second-highest valuation is

$$NF_{N-1}(v)^{N-1} - (N-1)F_{N-1}(v)^N.$$

Therefore,

$$\mathbb{E}_{F_{N-1}} [v_{(2)}] = \bar{v} - \int_{\underline{v}}^{\bar{v}} [NF_{N-1}(v)^{N-1} - (N-1)F_{N-1}(v)^N] \, dv.$$

Subtracting the two expressions gives

$$\begin{aligned}R^{\text{SPA}} - \mathbb{E}_{F_{N-1}} [v_{(2)}] &= \int_{\underline{v}}^{\bar{v}} \left[NF_{N-1}(v)^{N-1} - (N-1)F_{N-1}(v)^N \right. \\ &\quad \left. - N \int_{\Theta} F_\theta(v)^{N-1} \mu(d\theta) + (N-1) \int_{\Theta} F_\theta(v)^N \mu(d\theta) \right] \, dv.\end{aligned}$$

By the definition of F_{N-1} ,

$$F_{N-1}(v)^{N-1} = \int_{\Theta} F_\theta(v)^{N-1} \mu(d\theta).$$

Hence the two terms multiplied by N cancel, and we obtain

$$R^{\text{SPA}} - \mathbb{E}_{F_{N-1}} [v_{(2)}] = (N-1) \int_{\underline{v}}^{\bar{v}} \left[\int_{\Theta} F_\theta(v)^N \mu(d\theta) - F_{N-1}(v)^N \right] \, dv.$$

We next calculate the difference between the two FPA revenues. Applying integration by parts for Lebesgue–Stieltjes integrals,

$$\begin{aligned}
& \int_V \beta_0(v) d(F_{N-1}(v)^N) - \int_V \beta_0(v) d\left(\int_{\Theta} F_{\theta}(v)^N \mu(d\theta)\right) \\
&= - \int_V \beta_0(v) d\left[\int_{\Theta} F_{\theta}(v)^N \mu(d\theta) - F_{N-1}(v)^N\right] \\
&= - \beta_0(v) \left[\int_{\Theta} F_{\theta}(v)^N \mu(d\theta) - F_{N-1}(v)^N\right] \Big|_{\underline{v}}^{\bar{v}} \\
&\quad + \int_V \left[\int_{\Theta} F_{\theta}(v)^N \mu(d\theta) - F_{N-1}(v)^N\right] d\beta_0(v).
\end{aligned}$$

The boundary term vanishes: at $\underline{v}$, both CDFs are zero, while at $\bar{v}$, both are one. Hence

$$\begin{aligned}
& \int_V \beta_0(v) d(F_{N-1}(v)^N) - \int_V \beta_0(v) d\left(\int_{\Theta} F_{\theta}(v)^N \mu(d\theta)\right) \\
&= \int_V \left[\int_{\Theta} F_{\theta}(v)^N \mu(d\theta) - F_{N-1}(v)^N\right] d\beta_0(v).
\end{aligned}$$

Combining the two calculations gives

$$\begin{aligned}
R^{\text{SPA}} - R_{\text{id}}^{\text{FPA}} &= (N-1) \int_{\underline{v}}^{\bar{v}} \left[\int_{\Theta} F_{\theta}(v)^N \mu(d\theta) - F_{N-1}(v)^N\right] dv \\
&\quad + \int_V \left[\int_{\Theta} F_{\theta}(v)^N \mu(d\theta) - F_{N-1}(v)^N\right] d\beta_0(v).
\end{aligned} \tag{24}$$

For every $v \in V$, monotonicity of L^p norms implies

$$\int_{\Theta} F_{\theta}(v)^N \mu(d\theta) \geq \left(\int_{\Theta} F_{\theta}(v)^{N-1} \mu(d\theta)\right)^{N/(N-1)} = F_{N-1}(v)^N.$$

Both terms on the right-hand side of [Eq. \(24\)](#) are therefore nonnegative, because β_0 is non-decreasing.

Finally, the maintained nondegeneracy of $\mathcal{F}$, joint continuity, and $\text{supp } \mu = \Theta$ imply that $\theta \mapsto F_{\theta}(v)$ is not μ -almost surely constant for every v in some nondegenerate interval contained in V° . The strict L^p -norm inequality holds throughout this interval, so the first term on the right-hand side of [Eq. \(24\)](#) is strictly positive. Consequently,

$$R_{\text{id}}^{\text{FPA}} < R^{\text{SPA}}.$$

□

Proof of Proposition 4. By Theorem 3, we have

$$\beta_\ell(v) \longrightarrow \beta^{\min}(v) \quad \text{for every } v \in V,$$

where $\beta^{\min} = \beta_{\{F^+\}}$ is the standard risk-neutral FPA bid associated with the most competitive endpoint distribution F^+ . Since $0 \leq \beta_\ell(v) \leq v \leq \bar{v}$, dominated convergence implies

$$\lim_{\ell \rightarrow \infty} R_{\phi_\ell}^{\text{FPA}} = \int_{\Theta} \int_V \beta^{\min}(v) d(F_\theta(v)^N) \mu(d\theta). \quad (25)$$

The predictive SPA revenue is given by

$$R^{\text{SPA}} = \int_{\Theta} \int_V \beta_{\{F_\theta\}}(v) d(F_\theta(v)^N) \mu(d\theta). \quad (26)$$

Combining Eq. (25) and Eq. (26) yields

$$\lim_{\ell \rightarrow \infty} (R_{\phi_\ell}^{\text{FPA}} - R^{\text{SPA}}) = \int_{\Theta} \int_V [\beta^{\min}(v) - \beta_{\{F_\theta\}}(v)] d(F_\theta(v)^N) \mu(d\theta),$$

The limiting integrand is weakly nonnegative because the most competitive endpoint distribution F^+ has the highest reverse hazard rate in $\mathcal{F}$ and therefore induces the highest singleton-prior equilibrium bid. It follows that the preceding limit is weakly nonnegative.

Since $\mathcal{F}$ is nondegenerate and μ has full support, the inequality is strict.

$$\lim_{\ell \rightarrow \infty} (R_{\phi_\ell}^{\text{FPA}} - R^{\text{SPA}}) > 0.$$

Since the revenue difference converges to a strictly positive number, there exists $\ell_0 < \infty$ such that

$$R_{\phi_\ell}^{\text{FPA}} - R^{\text{SPA}} > 0 \quad \text{for every } \ell \geq \ell_0.$$

This proves the result. $\square$

Proof of Proposition 5. If $\theta' \geq \theta$, the RHR order implies that $F_{\theta'}$ first-order stochastically dominates F_θ . Since the ambiguity-equilibrium bid β is strictly increasing,

$$R^{\text{FPA}}(F_{\theta'}) \geq R^{\text{FPA}}(F_\theta).$$

The same stochastic order raises the second-highest valuation, so

$$R^{\text{SPA}}(F_{\theta'}) \geq R^{\text{SPA}}(F_\theta).$$

Both inequalities are strict whenever $F_{\theta'} \neq F_{\theta}$.

It remains to establish the endpoint rankings. Since $\mathcal{F}$ is nondegenerate, $F^- \neq F^+$, and hence their reverse hazard rates differ on a nondegenerate interval. The ambiguity-adjusted reverse hazard rate in the equilibrium ODE is a positive-weight average of the candidate reverse hazard rates. Joint continuity and $\text{supp } \mu = \Theta$ therefore imply that this average lies strictly between the endpoint reverse hazard rates on a nondegenerate interval. The strict comparison argument for the bid ODE then yields

$$\beta_{\{F^-\}}(v) < \beta(v) < \beta_{\{F^+\}}(v)$$

on a set of valuations of positive measure.

Conditional revenue equivalence and the positivity of the candidate densities consequently give

$$R^{\text{FPA}}(F^-) = \int_V \beta(v) d(F^-(v)^N) > \int_V \beta_{\{F^-\}}(v) d(F^-(v)^N) = R^{\text{SPA}}(F^-),$$

and, analogously,

$$R^{\text{FPA}}(F^+) < R^{\text{SPA}}(F^+).$$

Continuity of the two revenue profiles then implies an interior intersection. $\square$

Proof of Corollary 2. By Proposition 5, both revenue profiles are weakly increasing, so their lower and upper endpoints are attained at F^- and F^+ , respectively. The strict endpoint comparisons in Proposition 5 give the result. $\square$

Appendix E RHR-Ordered Distribution Families

The following examples concern only the RHR-ordering requirement. To fit the bounded-support baseline environment, each family may be truncated at a common finite upper endpoint; such a truncation leaves its RHR unchanged on the truncated support.

Proposition 8. *Let*

$$\mathcal{F} = \{ F_s : X \rightarrow [0, 1] \mid s \in S \subset (0, \infty) \},$$

where $X \subseteq \mathbb{R}$. Each of the following one-parameter families is totally ordered by the RHR order. After orienting the scalar index so that a larger index corresponds to a higher reverse hazard rate, each family satisfies the ordering requirement in Assumption 1.

1. **Exponential** with rate s :

$$F_s(x) = 1 - e^{-sx}, \quad x \geq 0.$$

2. **Pareto** with shape $s > 0$ and fixed scale $x_m > 0$:

$$F_s(x) = \begin{cases} 1 - (x_m/x)^s, & x \geq x_m, \\ 0, & x < x_m. \end{cases}$$

3. **Gamma** with shape $s > 0$ and fixed scale $\theta > 0$:

$$F_s(x) = \frac{\gamma(s, \frac{x}{\theta})}{\Gamma(s)}, \quad x \geq 0.$$

4. **Chi and chi-squared** distributions with $s > 0$ degrees of freedom. If $X_s \sim \chi_s$, then

$$X_s^2 \sim \chi_s^2 \sim \text{Gamma}(s/2, 2).$$

5. **Half-Normal** with scale $s > 0$:

$$F_s(x) = \Pr(|N(0, s^2)| \leq x) = \text{erf}\left(\frac{x}{\sqrt{2}s}\right), \quad x \geq 0.$$

For $N \geq 2$, the corresponding highest-rival family $H_s = F_s^{N-1}$, with $h_s := H'_s$, inherits the same parameter ordering because

$$\frac{h_s(v)}{H_s(v)} = (N-1) \frac{f_s(v)}{F_s(v)}.$$

Proof. Fix two parameters $s_1 < s_2$. Let ℓ denote the common lower endpoint of the support: $\ell = 0$ for the Exponential, Gamma, Chi, chi-squared, and Half-Normal families, and $\ell = x_m$ for the Pareto family. All likelihood-ratio comparisons below are conducted on the common interior of the support. For $x > \ell$,

$$\begin{aligned} & f_{s_1}(x)F_{s_2}(x) - f_{s_2}(x)F_{s_1}(x) \\ &= \int_{\ell}^x [f_{s_1}(x)f_{s_2}(t) - f_{s_2}(x)f_{s_1}(t)] dt. \end{aligned}$$

The integrand can be written as

$$f_{s_1}(x)f_{s_1}(t) \left[\frac{f_{s_2}(t)}{f_{s_1}(t)} - \frac{f_{s_2}(x)}{f_{s_1}(x)} \right].$$

Hence the sign of the displayed difference is fixed across the common interior of the support, which establishes comparability in the RHR order.⁴

(1) Exponential. Here

$$\frac{f_{s_2}(t)}{f_{s_1}(t)} = \frac{s_2}{s_1} e^{-(s_2-s_1)t}$$

is strictly decreasing in $t > 0$.

(2) Pareto. For $t > x_m$,

$$\frac{f_{s_2}(t)}{f_{s_1}(t)} = \frac{s_2}{s_1} \left(\frac{x_m}{t} \right)^{s_2-s_1},$$

which is strictly decreasing in t .

(3) Gamma. For fixed scale θ ,

$$\frac{f_{s_2}(t)}{f_{s_1}(t)} = \frac{\Gamma(s_1)}{\Gamma(s_2)} \theta^{s_1-s_2} t^{s_2-s_1},$$

which is strictly increasing in $t > 0$.

(4) Chi and chi-squared. For the Chi family,

$$\frac{f_{s_2}^\chi(t)}{f_{s_1}^\chi(t)} = 2^{-(s_2-s_1)/2} \frac{\Gamma(s_1/2)}{\Gamma(s_2/2)} t^{s_2-s_1},$$

which is strictly increasing in $t > 0$ when $s_2 > s_1$. For the chi-squared family,

$$\frac{f_{s_2}^{\chi^2}(t)}{f_{s_1}^{\chi^2}(t)} = 2^{-(s_2-s_1)/2} \frac{\Gamma(s_1/2)}{\Gamma(s_2/2)} t^{(s_2-s_1)/2},$$

which is also strictly increasing in $t > 0$.

(5) Half-Normal. For $s_1 < s_2$,

$$\frac{f_{s_2}(t)}{f_{s_1}(t)} = \frac{s_1}{s_2} \exp \left\{ \frac{t^2}{2} \left(\frac{1}{s_1^2} - \frac{1}{s_2^2} \right) \right\}.$$

Since $\frac{1}{s_1^2} - \frac{1}{s_2^2} > 0$, this ratio is strictly increasing in $t > 0$.

⁴A likelihood ratio that is monotone in either direction implies comparability in the RHR order. The scalar parameter is reoriented when necessary to match the convention in [Assumption 1](#); see [Shaked and Shanthikumar \(2007\)](#).

For the Exponential rate and Pareto shape,

$$s_1 < s_2 \implies F_{s_1} \succeq_{\text{RHR}} F_{s_2}.$$

Thus the index must be reversed, for example by using $\theta = -s$, to match [Assumption 1](#). For the Gamma shape, Chi degrees of freedom, chi-squared degrees of freedom, and Half-Normal scale,

$$s_1 < s_2 \implies F_{s_2} \succeq_{\text{RHR}} F_{s_1}.$$

Consequently, each family satisfies the ordering requirement in [Assumption 1](#) after the appropriate reorientation. $\square$

References

- AUSTER, SARAH AND CHRISTIAN KELLNER (2022): “Robust Bidding and Revenue in Descending Price Auctions,” *Journal of Economic Theory*, 199, 105072.
- BAILLON, AURÉLIEN AND LÆTITIA PLACIDO (2019): “Testing Constant Absolute and Relative Ambiguity Aversion,” *Journal of Economic Theory*, 181, 309–332.
- CARVALHO, MIGUEL (2012): “Static vs. Dynamic Auctions with Ambiguity Averse Bidders,” Center Discussion Paper Series No. 2012-022, March 12, 2012. Available at SSRN: <https://ssrn.com/abstract=2025872> or <http://dx.doi.org/10.2139/ssrn.2025872>.
- CHEN, YAN, PETER KATUŠČÁK, AND EMRE OZDENOREN (2007): “Sealed Bid Auctions with Ambiguity: Theory and Experiments,” *Journal of Economic Theory*, 136, 513–535.
- FABBRI, FRANCESCO, GIULIO PRINCIPI, AND LORENZO STANCA (2024): “Absolute and Relative Ambiguity Attitudes,” arXiv:2406.01343.
- GILBOA, ITZHAK AND DAVID SCHMEIDLER (1989): “Maxmin Expected Utility with Non-Unique Prior,” *Journal of Mathematical Economics*, 18, 141–153.
- GRETSCHKO, VITALI AND HELENE MASS (2024): “Worst-Case Equilibria in First-Price Auctions,” *Theoretical Economics*, 19, 61–93.
- KASBERGER, BERNHARD AND KARL H. SCHLAG (2024): “Robust Bidding in First-Price Auctions: How to Bid without Knowing What Others Are Doing,” *Management Science*, 70, 4219–4235.
- KIM, KWANGHYUN (2024): “First-Price Auctions with Maxmin Expected Utility Bidders,” Manuscript, June 26, 2024.
- KLIBANOFF, PETER, MASSIMO MARINACCI, AND SUJOY MUKERJI (2005): “A Smooth Model of Decision Making under Ambiguity,” *Econometrica*, 73, 1849–1892.
- LO, KIN CHUNG (1998): “Sealed Bid Auctions with Uncertainty Averse Bidders,” *Economic Theory*, 12, 1–20.
- MASKIN, ERIC AND JOHN RILEY (1984): “Optimal Auctions with Risk Averse Buyers,” *Econometrica*, 52, 1473–1518.
- MATTHEWS, STEVEN (1987): “Comparing Auctions for Risk Averse Buyers: A Buyer’s Point of View,” *Econometrica*, 55, 633–646.

- RAYMOND, COLLIN AND YANGWEI SONG (2026): “Risk and Monotone Comparative Statics without Independence,” arXiv:2601.10664.
- SHAKED, MOSHE AND J. GEORGE SHANTHIKUMAR (2007): *Stochastic Orders*, Springer.
- SONG, YANGWEI (2025): “Robust Auctions with Affiliated Private Values,” *Review of Economic Design*, 29, 245–279.